International Journal for Housing Science and Its Applications
Publish August 3, 2025. Volume 46, Issue 3 Pages 1120-1134

https://doi.org/10.70517/ijhsa46387

Algorithms for solving nonlinear optimization problems based on generalized extended solutions and their stability analysis

Cong Li[1,*]
[1] School of Mathematics and Statistics, Xinyang College, Xinyang, Henan, 464000, China
Corresponding authors: (e-mail: licong358572@163.com).

Abstract In this paper, a hybrid approximation $F_{(fh\alpha)} = a_1 + a_2(fh\alpha) + a_3(fh\alpha)^2$ and prediction $\begin{cases} \min I(a_1, a_2, a_3) = \sum_{i=1}^{n} \left[a_1 + a_2 x_i + a_3 x_i^2 - y_i \right]^2 \\ s.t. a_1 + a_2 x_n + a_3 x_n^2 = y_n, i = 1, 2, L, n-1 \end{cases}$ model based on the generalized extension method is constructed for the optimization problem of nonlinear systems, and a Lyapunov stability analysis is carried out. The model is applied to the nonlinear gear micro-parameter optimization system and the trajectory tracking optimization system for optimal parameter solving and physical trajectory prediction reduction. During the gear microscopic parameter optimization process, most of the optimal parameter solving errors of the generalized extension method are between 0.00 and 0.02, which have high accuracy. The convex eigenfunction initial value construction and the spectral method discretization are utilized to solve the problem of efficiently solving the nonlinear algebraic equation system in the case of multiple eigenvalues, and the flight physical trajectory is effectively restored. The Liapunov stability analysis shows that the generalized prolongation method satisfies the spectral stability condition under subharmonic perturbation.

Index Terms generalized extension method, convex function, Liapunov stability, spectral method, nonlinear optimization

I. Introduction

In daily life, the problem of maximizing the benefit based on available resources or minimizing the cost to achieve an objective is called an optimization problem [1]. With the development of science and technology and economy, optimization problems have been widely used in the fields of military, transportation, engineering, economy, national defense, artificial intelligence and social sciences, etc., which have attracted extensive attention of scholars and become a more widely used discipline [2]-[5]. The numerical solution of nonlinear optimization problems, on the other hand, is an important branch in the study of optimization problems [6]. However, many problems in engineering applications such as signal processing, system identification, robot motion control, etc., usually contain time-varying parameters and thus must be solved in real time in order to optimize the performance of dynamic systems [7], [8]. Such real-time application problems place stringent demands on computational time, making the numerical methods described above less effective [9].

As the form of the problem becomes more and more complex and the computer technology progressively deepens, the requirements on the performance of the algorithms become higher and higher [10]. Therefore, while proposing new methods, exploring algorithms with fast convergence, high stability and wide practical applicability has become a major challenge in the recent past [11], [12]. In the future development, the application areas of algorithms will be wider, and the better and faster development of algorithms will bring more positive changes to people's daily life [13], [14]. Therefore, exploring the solution method based on the generalized extended solution provides an effective method for dealing with nonlinear optimization problems, which complements and improves the existing theories and methods.

In this paper, the generalized extension method is taken as the core, and the theory of convex function analysis and Liapunov stability judgment are integrated to realize the optimization of nonlinear problems. By constructing the convex function $h(\gamma r + (1-\gamma)s) \le \gamma h(r) + (1-\gamma)h(s)$ with orthogonal projection operator $P_{\mathrm{D}}(u) := \arg\min_{z \in \mathrm{D}} \|u - z\|^2$, combined with a local fit $U_{(FH\alpha)} = \sum_{j=1}^{t} a_j \Phi_{j(fh\alpha)}$ with extrapolation strategy

$\begin{cases} \min I(a_1,a_2,a_3)=\sum_{i=1}^{n}\left[a_1+a_2x_i+a_3x_i^2-y_i\right]^2 \\ s.t. a_1+a_2x_n+a_3x_n^2=y_n, i=1,2,L,n-1 \end{cases}$, realizing high-precision segmental approximation and dynamic sequence prediction by generalized extension. Solve the optimal value of nonlinear gear micro-parameters to improve the gear running accuracy. Predict and solve the flight trajectory to restore the physical traces. Utilizing Liapunov stability analysis to verify the solution stability.

II. Theoretical elaboration of the generalized extension approach

In this chapter, the optimization theory of the generalized extension method and the definitions and theorems related to matrix analysis are explained. Construct the generalized extension model for use in the optimization solution of nonlinear problems. Analyze the application value of Lyapunov stability in the optimization of nonlinear systems.

II. A. Optimization theory and matrix ana C lysis

Definition 1: Denote $h:\mathrm{C}\to(-\infty,+\infty]$ as a true function, where the convex set $\mathrm{C}\subseteq\mathbb{R}^m$ is the domain of definition (also known as the domain of validity) of the function $h(\cdot)$. For $\forall r,s\in\mathrm{C}$ and $\forall\gamma\in[0,1]$, the following inequality relation is true if

$$.\, h(\gamma r+(1-\gamma)s)\le\gamma h(r)+(1-\gamma)h(s). \tag{1}$$

always holds, then $h(\cdot)$ is said to be a convex function on . Moreover, if $-h(\cdot)$ is a convex function on C, then $h(\cdot)$ is said to be a concave function on . C..

Definition 2: Let $g:\mathrm{B}\to\mathbb{R}$ denote a real-valued truly convex function on the convex set $\mathrm{B}\subseteq\mathbb{R}^m$. Let $p\in\mathbb{R}^m$ be any point in the set B. For any $q\in\mathrm{B}$, if the m-dimensional real column vector r satisfies the following inequality relation

$$g(p)-g(q)\ge\langle r,p-q\rangle \tag{2}$$

Then $r\in\mathbb{R}^m$ is called a subgradient of the convex function $g(\cdot)$ at the point $p\in\mathrm{B}$.

Remark 1: If the convex function $g(\cdot)$ is derivable at a point, the gradient of $g(\cdot)$ at that point is the unique subgradient of $g(\cdot)$ at that point.

Definition 3: If $g(\cdot)$ consists of the real-valued truly convex function involved in Definition 2, then the set consisting of all subgradients of the convex function $g(\cdot)$ at the point $p\in\mathrm{B}$ is called the subdifferential of $g(\cdot)$ at the point $p\in\mathrm{B}$ (abbreviated as $\partial g(p)$), i.e.

$$\partial g(p):=\left\{r\in\mathbb{R}^m : g(p)-g(q)\ge\langle r,p-q\rangle,\forall q\in\mathrm{B}\right\}. \tag{3}$$

Definition 4: Let $\mathrm{D}\subseteq\mathbb{R}^m$ be a nonempty closed convex set. To this end, the orthogonal projection operator $P_{\mathrm{D}}(\cdot):\mathbb{R}^m\to\mathrm{D}$ on the set D is given by the following equation

$$P_{\mathrm{D}}(u):=\arg\min_{z\in\mathrm{D}}\left\|u-z\right\|^2 \tag{4}$$

Defined, where $\forall u\in\mathbb{R}^m$.

Remark 2: The orthogonal projections have succinct explicit formulas for the following special classes of convex sets.

(1) Denote $\mathbb{R}_+^m$ as an m-dimensional nonnegative real column vector space, i.e., for $\forall z\in\mathbb{R}_{\ge0}^m$, each component of the vector z is nonnegative real. For an arbitrary real number r, $[r]_+$ denotes its non-negative component, i.e.

$$\lfloor r\rfloor_+:=\begin{cases} r & r\ge0 \\ 0 & r<0 \end{cases} \tag{5}$$

For $\forall r=(r_1,r_2,\cdots,r_m)^{\cdot}\in\mathbb{R}^m$, notate $\lfloor r\rfloor_+:=\left(\lfloor r_1\rfloor_+,\lfloor r_2\rfloor_+,\cdots,\lfloor r_m\rfloor_+\right)^{\cdot}$. Then, the orthogonal projection on the nonnegative real column vector space $\mathbb{R}_{\ge0}^m$ is

$$P_{\mathbb{R}_{\ge0}^m}(r):=\lfloor r\rfloor_+,\forall r\in\mathbb{R}^m \tag{6}$$

(2) Let $r=\left(r_1,r_2,\cdots,r_m\right)^{\bullet}\in\mathbb{R}^m$. Remember that $\mathrm{D}\subseteq\mathbb{R}^m$ is a box set, i.e., $D=\Pi\left[\underline{b}_k,\overline{b}_k\right]=\left\{r\in\mathbb{R}^m:\underline{b}_k\le r_k\le\overline{b}_k,\forall k\in\lceil m\rceil\right\}$, here $\underline{b}_k\le\overline{b}_k$. Then, the orthogonal projection on the box set D is

$$s=\left(s_1,s_2,\cdots,s_m\right)^{\bullet}:=P_{\mathrm{D}}(r),\forall r\in\mathbb{R}^m \tag{7}$$

Among them:

$$s_k=\begin{cases}\overline{b}_k, & r_k\ge\overline{b}_k\\ r_k, & \underline{b}_k<r_k<\overline{b}_k\\ \underline{b}_k, & r\le\underline{b}_k\end{cases},\forall k\in\lceil m\rceil \tag{8}$$

(3) Remember that $\mathrm{D}\subseteq\mathbb{R}^m$ is the set of ℓ_2-paradigm spheres of radius $s\in\mathbb{R}_+$ centered at the origin, i.e., $\mathrm{D}:=\left\{r\in\mathbb{R}^m:\|r\|\le s\right\}$. Then, the orthogonal projection on the ℓ_2-paradigm spherical set D is

$$P_{\mathrm{D}}(r):=\begin{cases}r & \|r\|\le s\\ \dfrac{sr}{\|r\|} & \|r\|>s\end{cases}\forall r\in\mathbb{R}^m \tag{9}$$

(4) Remember that $\mathrm{D}\subseteq\mathbb{R}^m$ is a hyperplane, i.e., $\mathrm{D}:=\left\{r\in\mathbb{R}^m:\langle a,r\rangle=h,a\in\mathbb{R}^m,h\in\mathbb{R}\right\}$, where $a\ne 0$. Then, the orthogonal projection on the hyperplane D is

$$P_{\mathrm{D}}(r):=r+\frac{\left(h-\langle a,r\rangle\right)a}{\|a\|^2},\forall r\in\mathbb{R}^m \tag{10}$$

(5) Remember that $\mathrm{D}\subseteq\mathbb{R}^m$ is a half-space, i.e., $\mathrm{D}:=\left\{r\in\mathbb{R}^m:\langle a,r\rangle\le h,a\in\mathbb{R}^m,h\in\mathbb{R}\right\}$, where $a\ne 0$. Then, the orthogonal projection on the half-space D is

$$P_{\mathrm{D}}(r):=\begin{cases}r+\dfrac{(h-\langle a,r\rangle)a}{\|a\|^2} & \langle a,r\rangle>h\\ r & \langle a,r\rangle\le h\end{cases}\forall r\in\mathbb{R}^m \tag{11}$$

(6) Remember that $\mathrm{D}\subseteq\mathbb{R}^m$ is the affine set, i.e., $\mathrm{D}:=\left\{r\in\mathbb{R}^m:Hr-h=0,H\in\mathbb{R}^{p\times m},h\in\mathbb{R}^p\right\}$, where $rank\left(H\right)=p$. Then, the orthogonal projection on the affine set D is

$$P_{\mathrm{D}}(r):=r-H^{\bullet}\left(HH^{\bullet}\right)^{-1}\left(Hr-h\right),\forall r\in\mathbb{R}^m \tag{12}$$

If $HH^{\bullet}=E$ or $p\ll m$, then the orthogonal projection $P_{\mathrm{D}}(\cdot)$ is quite inexpensive to compute.

II. B. Modeling

II. B. 1) Generalized Extended Approximation Model Construction

Using the method of generalized extended approximation to $fh\alpha$, for example, in order to ensure that the approximation function on the domain of any cell is continuous and smooth with the function on the domain of the surrounding cells, the region $\Omega_{(fh\alpha)}$ composed of the revision quantity $fh\alpha$ and the discrete points of the noise vibration data is partitioned into m mutually non-overlapping subdomains $\Omega_{(fh\alpha)e}$ and its domain of definition of interest can be extended to neighboring cells $\Omega_{(fh\alpha)e'}$ with $\Omega_{(fh\alpha)}=U_{e=1}^{m}\Omega_{(fh\alpha)e}$, and $\Omega_{(fh\alpha)e}\in\Omega_{(fh\alpha)e'}$. Here assume that there are q nodes inside $\Omega_{(fh\alpha)e'}$, and there are s nodes belonging to the category of $\Omega_{(fh\alpha)e}\left(s<q\right)$, and utilize the localized fitting approximation algorithm to construct the prolonged approximation function $U_{(FH\alpha)}=\sum_{j=1}^{t}a_j\Phi_{j(fh\alpha)}$, $s<t<q$, where $\Phi_{1(fh\alpha)}$, $\Phi_{2(fh\alpha)}$, $\cdots$, and $\Phi_{t(fh\alpha)}$ are $\Omega_{(fh\alpha)e'}$ on the basis functions. Let the extended approximation function on $\Omega_{(fh\alpha)e'}$ be quadratic, then 1, $fh\alpha$, $fh\alpha^2$ are a set of basis functions, and the relationship between vibrational noise data and the amount of revision can be expressed as

$$F_{(fh\alpha)} = a_1 + a_2(fh\alpha) + a_3(fh\alpha)^2 \tag{13}$$

a_1, a_2, a_3 are the coefficients to be determined.

The best squared approximation of $F_{(fh\alpha)}$ using node $fh\alpha_{(i)}(i=1,2,3,\cdots,n)$ within $\Omega_{(fh\alpha)e'}$, while enabling $F_{(fh\alpha)}$ to satisfy the difference condition on $\Omega_{(fh\alpha)e}$, which in turn makes the constructor have the minimum squared approximation error, i.e:

$$\begin{aligned} &\min I(a_1,a_2,a_3) = \sum_{i=1}^{n}(a_1 + a_2 * fh\alpha_i + a_3 * fh\alpha_i^2 - F_i)^2 \\ &S.t.\ a_1 + a_2 * fh\alpha_1 + a_2 * fh\alpha_1^2 = F_1 \\ &a_1 + a_2 * fh\alpha_2 + a_2 * fh\alpha_2^2 = F_2 \\ &a_1 + a_2 * fh\alpha_j + a_2 * fh\alpha_j^2 = F_j \end{aligned} \tag{14}$$

Using the Lagrange multiplier method to solve the above equation with Lagrange multiplier $\lambda_1,\lambda_2,\cdots,\lambda_j$, the constructor is as follows:

$$\begin{aligned} &L(a_1,a_2,a_3,\lambda_1,\lambda_2,\cdots,\lambda_j) = I(a_1,a_2,a_3) \\ &+\lambda_1(a_1 + a_2 * fh\alpha_1 + a_3 * fh\alpha_1^2 - F_1) \\ &+\lambda_2(a_1 + a_2 * fh\alpha_2 + a_3 * fh\alpha_2^2 - F_2) \\ &+\lambda_j(a_1 + a_2 * fh\alpha_j + a_3 * fh\alpha_j^2 - F_j) \end{aligned} \tag{15}$$

From the practical point of view, considering the cost of testing and computational efficiency, n in Eq. (15) is taken to be 5 and j is taken to be $\{(fh\alpha_1,F_1),(fh\alpha_2,F_2),\cdots,(fh\alpha_n,F_n)\}$. The test discrete point 3 in the selected sub-domain can satisfy the demand, which is obtained from $\partial L/\partial a_i = 0(i=1,2,3)$, $\partial L/\partial\lambda_j = 0(j=1,2)$:

$$\begin{bmatrix} 4 & \sum_{i=1}^{4} fh\alpha_i & \sum_{i=1}^{4} fh\alpha_i^2 & 1 & 1 \\ \sum_{i=1}^{4} fh\alpha_i & \sum_{i=1}^{4} fh\alpha_i^2 & \sum_{i=1}^{4} fh\alpha_i^3 & fh\alpha_1 & fh\alpha_2 \\ \sum_{i=1}^{4} fh\alpha_i^2 & \sum_{i=1}^{4} fh\alpha_i^3 & \sum_{i=1}^{4} fh\alpha_i^4 & fh\alpha_1^2 & fh\alpha_2^2 \\ 1 & fh\alpha_1 & fh\alpha_1^2 & 0 & 0 \\ 1 & fh\alpha_2 & fh\alpha_2^2 & 0 & 0 \end{bmatrix} \times \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{4} F_i \\ \sum_{i=1}^{4} F_i \\ \sum_{i=1}^{4} F_i \\ F_1 \\ F_2 \end{pmatrix} \tag{16}$$

Based on Eq. (16), the segmental approximation function $F_{(fh\alpha)}$ in Eq. (13) can be obtained by bringing in the measured discrete-point data, and the interpolation is computed by taking the nodes equally spaced on the interval function, and then constructing the approximation function $F_{i(fh\alpha)}$ on each interval sequentially, and then all the performance curves are fitted after all the interpolated points have been found out, and similarly, the performance curves can be obtained by finding out the value of $F_{(fh\beta)}$.

II. B. 2) Generalized Extended Prediction Model Construction

For the growing sequence (y_1,x_1), L, (y_n,x_n), L, follow the design concept of generalized delayed interpolation extrapolation, so that x_n be the latest moment, and model the generalized delayed extrapolation as:

$$\begin{cases} \min I(a_1,a_2,a_3) = \sum_{i=1}^{n}\left[a_1 + a_2x_i + a_3x_i^2 - y_i\right]^2 \\ s.t. a_1 + a_2x_n + a_3x_n^2 = y_n, i = 1,2,L,n-1 \end{cases} \tag{17}$$

Generalized extended prediction models can be solved using the Lagrange multiplier method. When fitting with a priori data points, the fitted points can be multiplied by a weighting factor, and the weighting factor can be applied in such a way that the newer the data point, the larger the weighting factor. The newest data points can also be processed in a variety of ways when utilized, such as a number of the newest data points can be averaged and coalesced into a single interpolated point for processing.

II. C.Lyapunov stabilization

Stability is a basic structural characteristic of a control system, which indicates the ability of a system to maintain its preset working state after being perturbed. Stability is also a prerequisite for a system to be able to operate properly, and only a stable system can be expanded to practical applications. Liapunov stability theory plays a very important role in the study of automatic control systems, and because of its universal and intuitive characteristics, it is popular in the stability analysis of nonlinear system control, and even in the stability analysis of other control systems has been widely used. In this paper, the control method of strict feedback nonlinear system is investigated based on the Liapunov stability theorem, and the following are the relevant definitions of Liapunov stability.

Definition 1 (Stability in the sense of Liapunov):

(1) For a nonlinear system $\dot{x} = f(t,x), x(t_0) = x_0, t \in [t_0, \infty)$ in an isolated equilibrium state x_e, if for any real number $\varepsilon > 0$, there is $\delta(\varepsilon, t_0) > 0$ such that the inequality $\|x_0 - x_e\| \le \delta(\varepsilon, t_0)$ of the perturbed motion departing from any initial state x_0 satisfies $\|\phi(t; x_0, t_0) - x_e\| \le \varepsilon, \forall t \ge t_0$, then x_e is stable in the Liapunovian sense at the moment t_0.

(2) If the system is Liapunov stable (i.e., the system is stable in the sense of satisfying 1)), for $\delta(\varepsilon, t_0)$ and any real number $\mu > 0$, there exists correspondingly a real number $T(\mu, \delta, t_0) > 0$ such that satisfies $\|x_0 - x_e\| < \delta(\varepsilon, t_0)$ of the perturbed motion $\phi(t; x_0, t_0)$ departing from any initial state x_0 can also satisfy the inequality: $\|\phi(t; x_0, t_0) - x_e\| \le \mu, \forall t \ge t_0 + T(\mu, \delta, t_0)$ then x_e is asymptotically stable at the moment t_0.

Theorem 1 (Lyapunov's stability principle): for a nonlinear system $\dot{x} = f(t,x), t \in [t_0, \infty)$, if the origin is an equilibrium point of the system, and let one of the neighborhoods of the origin be B, if there exists for all nonzero states $x \in B$ a positive definite function $V(x,t)$ satisfying $\dot{V}(x,t) \le 0$, then the system origin is said to be uniformly stable in the neighborhood B in the sense of Liapunov.

III. Generalized Extension Method Solution and Stability Analysis

This chapter analytically demonstrates the optimization process of nonlinear problems based on the generalized prolonged solution through the generalized prolonged solution of 2 nonlinear optimization systems. It also verifies the stability of the generalized extension solution process and results of the 2 nonlinear optimization systems through Liapunov stability analysis.

III. A. Generalized Extended Solution for Nonlinear Gear Microparameter Optimization Systems

In the nonlinear gear micro-parameter optimization system, the accurate operation of gears requires the simultaneous adjustment of multiple parameters. Constructing a one-dimensional function for each parameter and solving it using the generalized prolonged approximation method, the optimal solution is found to adjust and optimize the parameters, so as to reduce the cost of gear operation and improve the operation accuracy. At the same time, in order to judge the approximation effect of the generalized extension approximation method and to understand the error between the solution of the method and the actual optimal solution, the cubic spline interpolation method and linear interpolation method are chosen as the comparison methods to analyze the approximation error of the three methods under the parameter one-dimensional function solving.

Taking the unitary function related to the helix angle parameter that affects the direction of gear operation as an example, it is known that the function $f(x)$ has values $U_i = f(x_i)\left(i = 0,1,\cdots,n\right)$ at $n+1$ nodes in $[a,b]$ that are mutually exclusive, and it is desirable to find an approximation function $U(x)$ that satisfies $U(x_i) = U_i\left(i = 0,1,2,\cdots,n\right)$.

Follow the generalized extended approximation method for the algorithm. Let $\Delta = [a,b]$ and the node x_i satisfies $a = x_0 < x_1 < \cdots < x_n = b$, and so a partition of Δ can be obtained: $\Delta = \Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_n$ where $\Delta_e = [x_{e-1}, x_e]$, $e = 1,2,\cdots,n$, obtained by extending Δ_e (note: the extended nodes at both ends have to be processed).

$$\Delta'_e = \left\{x \middle| x \in [x_{e-2}, x_{e+1}]\right\}, e = 1,2,\cdots,n \tag{18}$$

Construct the segmented interpolation function on Δ_e. For the sake of uniformity, the nodes within Δ_e and Δ'_e are denoted $x_i^e\left(i=0,1,2,3\right)$, and the corresponding function values are $U_i^e\left(i=0,1,2,3\right)$.

Let the generalized interpolation function $U(x)$ within Δ_e be

$$U(x)=a_1+a_2x+a_3x^2, x\in\left[x_1^e, x_2^e\right] \tag{19}$$

where a_1, a_2, a_3 are coefficients to be determined by the following problem

$$\begin{cases} \min I\left(a_1,a_2,a_3\right)=\min\sum_{i=0}^{3}\left[a_1+a_2x_i^e+a_3\left(x_i^e\right)^2-U_i^e\right]^2 \\ s.t.\begin{cases} a_1+a_2x_1^e+a_3\left(x_1^e\right)^2=U_1^e \\ a_1+a_2x_2^e+a_3\left(x_2^e\right)^2=U_2^2 \end{cases} \end{cases} \tag{20}$$

Solving equation (20) yields a set of algebraic equations for a_1, a_2, a_3

$$\begin{bmatrix} 1 & x_1^e & (x_1^e)^2 \\ 1 & x_2^e & (x_2^e)^2 \\ \sum_{i=0}^{3}C_i & \sum_{i=0}^{3}C_ix_i^e & \sum_{i=0}^{3}C_i(x_i^e)^2 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}=\begin{bmatrix} U_1^e \\ U_2^e \\ \sum_{i=0}^{3}C_iU_i^e \end{bmatrix} \tag{21}$$

where $C_i=(x_i^e)^2-(x_1^e+x_2^e)x_i^e+x_1^ex_2^e$.

If this segmented interval lies at the leftmost end of the entire domain of definition, x_0 is bounded, i.e., let $x_0=x_1$; if this segmented interval lies at the rightmost end of the entire domain of definition, x_3 is bounded as $x_3=x_2$. It can be introduced that the generalized extended approximation function on these two small intervals is the same as the approximation function on this interval using the segmented quadratic interpolation approximation method.

The a_1^e, a_2^e, a_3^e are derived from Eq. (21) as approximate solutions to Eq. (21). The quadratic interpolation function on Δ_e is then determined. Constructing the generalized interpolating function on each interval in turn gives the generalized interpolating function for the segmental approximation on the full domain.

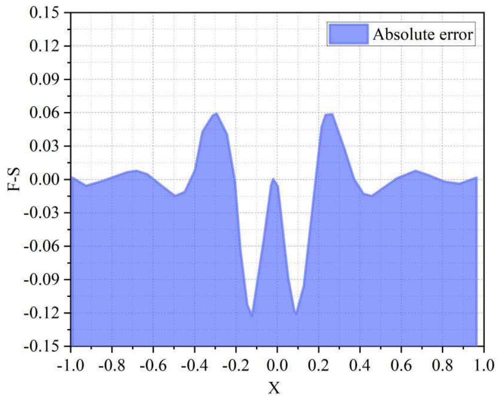
Given the function $f(x)=1/(1+50x^2)$, $x\in[-1,1]$, take the equidistant node $x=-1+i/5$, $i=0,1,2,\cdots,10$, the following approximation of $f(x)$ is made by the cubic spline interpolation method, linear interpolation, and generalized prolonged approximation method, respectively, and the error plots of the three approximation methods are drawn.

Figure 1 shows the approximation errors of the three methods. The fitted curve of the generalized extended approximation method approximates the original function curve the best. And as can be seen from Fig. 1, the cubic spline interpolation error is between [-0.13,0.06], the linear interpolation error is between [-0.06,0.10], and the generalized prolonged approximation method error is between [-0.12,0.05]. A closer look at Fig. 1 shows that with the change of x-value, most of the errors of the generalized delayed approximation method are between 0.00-0.02, while most of the error values of the other 2 methods fluctuate greatly. It shows that the error of the generalized delayed approximation method is the smallest and the error is stable. Therefore, the generalized extension approximation method is better than the cubic spline interpolation method and linear interpolation method. In addition, this method has a simple format specification, does not need to increase the type and scale of degrees of freedom, and can improve the approximation accuracy by using the original nodes and degrees of freedom, so it can be widely used in gear micro-parameter optimization system for optimal solving of various types of parameters.

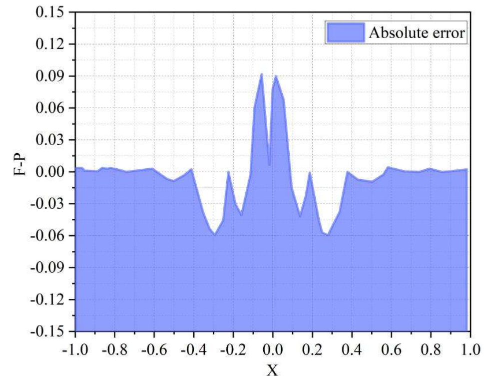
III. B. Generalized extension solution for nonlinear trajectory tracking optimization systems

The application of generalized extended solver methods to vehicle attitude control, trajectory tracking, and propulsion system control in aerospace can achieve accurate solutions and improve the performance and flight safety of the vehicle. In this section, one of the trajectory tracking systems is selected for flight trajectory tracking solution to maximize the accuracy of physical trajectory prediction for the flight process.

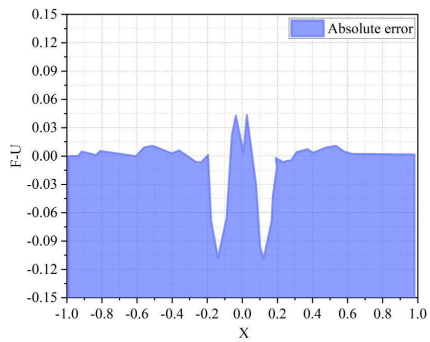
The initial values of the flight trajectory features have a large impact on the solution accuracy, so the initial values of the feature pairs are first constructed using convex functions. In the following, we first consider the two-dimensional resonator potential well case on the full space $\square^2$, which is realized as follows:



(a) Cubic spline interpolation error



(b) Linear interpolation error



(c) Error of this method

Figure 1: Three methods approximate the error

Step1: Calculate the eigenpairs of the operator $\left(-\Delta+V(x)\right)$

Consider the linear eigenvalue problem on full space

$$\begin{cases}\left(-\Delta+x^2+y^2\right)\Phi=\lambda\Phi, in\ \Omega=\mathbb{R}^2 \\ \|\Phi\|_{L^2(\Omega)}^2=1\end{cases} \tag{22}$$

Separate variables are used so that $\Phi(x,y)=\phi(x)\psi(y)$. Since the convex function satisfies

$$\hat{H}_{n''}(x)+(2n+1-x^2)\hat{H}_n(x)=0,\int_{-\infty}^{+\infty}\hat{H}_n(x)\hat{H}_m(x)dx=\delta_{nm} \tag{23}$$

The (22) characteristic pair can be found:

$$\lambda_{ij}=2(i+j+1),\Phi_{ij}(x,y)=\hat{H}_i(x)\hat{H}_j(y),i,j\in\mathbb{N} \tag{24}$$

It is clear that $\left(\Phi_{ij}(x,y),\Phi_{lk}(x,y)\right)=\delta_{il}\delta_{jk}$, i.e., the eigenfunctions are orthogonal.

The three-term recursive formula for the convex function is:

$$\begin{cases}\hat{H}_0(x)=\pi^{-\frac{1}{4}}e^{-\frac{x^2}{2}},\hat{H}_1(x)=\sqrt{2}\pi^{-\frac{1}{4}}xe^{-\frac{x^2}{2}} \\ \hat{H}_{n+1}(x)=x\sqrt{\frac{2}{n+1}}\hat{H}_n(x)-\sqrt{\frac{n}{n+1}}\hat{H}_{n-1}(x),n\geq 1\end{cases} \tag{25}$$

If λ is an m-heavy eigenvalue of (22), then there exist m pairs of integers $(i_p,j_p)\in\mathbb{N}\times\mathbb{N}$ with $p=0,1,\cdots,m-1$ such that

$$\lambda = 2(i_0 + j_0 + 1) = 2(i_1 + j_1 + 1) = \cdots = 2(i_{m-1} + j_{m-1} + 1) \tag{26}$$

Denote the orthogonal eigenfunctions corresponding to λ as

$$\Phi_p^\lambda(x,y) = \hat{H}_{i_p}(x)\hat{H}_{j_p}(y), \quad p = 0,1,\cdots,m-1 \tag{27}$$

Below we consider generating the subspace S_λ from the eigenfunctions $\{\Phi_p^\lambda\}$ corresponding to the same eigenvalue λ.

Step2: Search for initial values in a suitable subspace S_λ

Define $S_\lambda = span\{\Phi_0^\lambda, \Phi_1^\lambda, \cdots, \Phi_{m-1}^\lambda\}$, and consider the rough approximate solution $((u_\lambda, \mu_\lambda) \in S_\lambda \times \mathbb{R})$ to make the

$$\begin{cases} (\nabla u_\lambda, \nabla\varphi) + (V(x)u_\lambda, \varphi) + \beta(u_\lambda^3, \varphi) = \mu_\lambda(u_\lambda, \varphi), \forall\varphi \in S_\lambda \\ \|u_\lambda\|_{L^2}^2 = 1 \end{cases} \tag{28}$$

Taking $\varphi = \Phi_q^\lambda\,(q = 0,1,\cdots,m-1)$ and substituting $u_\lambda = \sum_{p=0}^{m-1} a_p \Phi_p^\lambda$ Substituting into equation (28) yields that

$$\begin{cases} \sum_{p=0}^{m-1} a_p \left((-\Delta + V(x))\Phi_p^\lambda, \Phi_q^\lambda\right) + \beta\left(\left(\sum_{p=0}^{m-1} a_p \Phi_p^\lambda\right)^3, \Phi_q^\lambda\right) \\ = \mu_\lambda\left(\sum_{p=0}^{m-1} a_p \Phi_p^\lambda, \Phi_q^\lambda\right) \\ \left\|\sum_{p=0}^{m-1} a_p \Phi_p^\lambda\right\|_{L^2}^2 = 1 \end{cases} \tag{29}$$

Using $\{\Phi_q^\lambda\}$ as eigenfunctions and orthogonal to each other, there are

$$\begin{aligned} &\sum_{p=0}^{m-1} a_p \left(\left(-\Delta + V(x)\right)\Phi_p^\lambda, \Phi_q^\lambda\right) = \sum_{p=0}^{m-1} a_p \left(\lambda\Phi_p^\lambda, \Phi_q^\lambda\right) \\ &= \lambda a_q, \quad q = 0,1,\cdots,m-1 \left\|\sum_{p=0}^{m-1} a_p \Phi_p^\lambda\right\|_{L^2}^2 \\ &= \sum_{p=0}^{m-1} a_p^2 \left\|\Phi_p^\lambda\right\|_{L^2}^2 = 1 \end{aligned} \tag{30}$$

In turn, equation (29) transforms into a system of nonlinear algebraic equations about $(a_0, \cdots, a_{m-1}, \mu_\lambda) \in \mathbb{R}^{m+1}$

$$\begin{cases} \lambda a_q + \beta\left(\left(\sum_{p=0}^{m-1} a_p \Phi_p^\lambda\right)^3, \Phi_q^\lambda\right) = \mu_\lambda a_q, q = 0,1,\cdots,m-1 \\ \sum_{p=0}^{m-1} a_p^2 = 1 \end{cases} \tag{31}$$

For the integral term in equation (31), numerical integration and other methods can be used. Solving (31) for the case where λ is a single, dual, and triple eigenvalue yields the following results:

(1) When $\lambda = \lambda_{00}$ (the unique unary eigenvalue) is a single eigenroot, i.e., $m = 1$, the

$$\begin{cases} \lambda_{00} a_{00} + \beta a_{00}^3 \left(\int_{-\infty}^{+\infty} \hat{H}_0^4(x)dx\right)^2 = \mu a_{00} \\ a_{00}^2 = 1 \end{cases} \tag{32}$$

The solution to the equation is found to be $a = \pm 1$ and $\mu = 2 + 0.1592\beta$.

(2) When $\lambda\left(\lambda = \lambda_{10} = \lambda_{01}\right)$ is a double root, i.e., when $m = 2$, the

$$\begin{cases}\lambda_{01}a_{01}+0.1195\beta a_{01}=\mu a_{01}\\ \lambda_{10}a_{10}+0.1195\beta a_{10}=\mu a_{10}\\ a_{01}^2+a_{10}^2=1\end{cases} \tag{33}$$

Solve for $a=(\pm1,0),(0,\pm1),\left(\frac{1}{\sqrt{2}},\pm\frac{1}{\sqrt{2}}\right),\left(-\frac{1}{\sqrt{2}},\pm\frac{1}{\sqrt{2}},\right)$, $\mu=4+0.1195\beta$.

(3) When $\lambda(\lambda=\lambda_{20}=\lambda_{11}=\lambda_{02})$ is a triple root, i.e., when $m=3$, 14 solutions of equation (31) are obtained:

$$a=(0,\pm1,0),\mu=6+0.0896\beta \tag{34}$$

$$a=\left(\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\right),\mu=6+0.0797\beta \tag{35}$$

$$a=\left(-\frac{1}{\sqrt{2}},0,-\frac{1}{\sqrt{2}}\right),\mu=6+0.0797\beta \tag{36}$$

$$a=\left(-\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\right),\mu=6+0.0896\beta \tag{37}$$

$$a=\left(\frac{1}{\sqrt{2}},0,-\frac{1}{\sqrt{2}}\right),\mu=6+0.0896\beta \tag{38}$$

$$a=\left(\pm\frac{1}{\sqrt{3}},\pm\frac{1}{\sqrt{3}},\pm\frac{1}{\sqrt{3}}\right),\mu=6+0.0929\beta \tag{39}$$

Step3: Further approximation of the solution in a suitably large subspace S_n

To obtain a better initial value than Step2, take a suitably large subspace S_n such that $S_\lambda\subset S_n$.

Define $S_n=span\{\Phi_{pq},p,q=0,1,\cdots,n-1\}$ to be the subspace into which the eigenfunctions are tensored to seek $((u_n,\mu_n)\in S_n\times\mathbb{R})$ to make the

$$\begin{cases}(\nabla u_n,\nabla\varphi)+(V(x)u_n,\varphi)+\beta(u_n^3,\varphi)=\mu_n(u_n,\varphi),\forall\varphi\in S_n\\ \|u_n\|_{L^2}^2=1\end{cases} \tag{40}$$

Let $u_n=\sum_{p,q=0}^{n-1}a_{pq}\Phi_{pq}$, and take $\varphi=\Phi_{lk}(l,k=0,1,\cdots,n-1)$, and utilize $\{\Phi_{pq}\}$ as an eigenfunction and with canonical orthogonality, i.e.

$$\sum_{p,q=0}^{n-1}a_{pq}\left((-\Delta+V(x))\Phi_{pq},\Phi_{lk}\right)=\sum_{p,q=0}^{n-1}a_{pq}\left(\lambda_{pq}\Phi_{pq},\Phi_{lk}\right)=a_{lk}\lambda_{lk} \tag{41}$$

$$\left\|\sum_{p,q=0}^{n-1}a_{pq}\Phi_{pq}\right\|_{L^2}^2=\sum_{p,q=0}^{n-1}a_{pq}^2=1 \tag{42}$$

Thus equation (40) is transformed into a system of nonlinear algebraic equations about $(a,\mu_n)\in\mathbb{R}^{n^2+1}$

$$\begin{cases}(\lambda_{lk}-\mu_n)a_{lk}+\beta\left(\left(\sum_{p,q=0}^{n-1}a_{pq}\Phi_{pq}\right)^3,\Phi_{lk}\right)=0,l,k=0,1,\cdots,n-1\\ \sum_{p,q=0}^{n-1}a_{pq}^2=1\end{cases} \tag{43}$$

Using $\left(a^0, \mu_n^0\right)$ as the initial value, the generalized extension method is used to solve (43), where a^0 and μ_n^0 are obtained from step2.

For the integral term $\left(\left(\sum_{p,q=0}^{n-1} a_{pq}\Phi_{pq}\right)^3, \Phi_{lk}\right)$ in equation (43), use Gauss-Hermite (GH) numerical integration. Let the two-dimensional GH points and GH weights be $\left\{\left(x_e, y_f\right)\right\}_{e,f=0}^{N}$, $\left\{w_{ef}\right\}_{e,f=0}^{N}$, respectively. Noting that $u = \sum_{p,q=0}^{n-1} a_{pq}\Phi_{pq}$, we have:

$$\begin{aligned}\left(\left(\sum_{p,q=0}^{n-1} a_{pq}\Phi_{pq}\right)^3, \Phi_{lk}\right) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} u^3\Phi_{lk}\,dxdy \\ &\approx \sum_{e=0}^{N}\sum_{f=0}^{N} u^3(x_e, y_f)\Phi_{lk}(x_e, y_f)w_{ef}\end{aligned} \tag{44}$$

In order to calculate the Jacobi of (44), it is also necessary to calculate the

$$\begin{aligned}\left(u^2\Phi_{pq}, \Phi_{lk}\right) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} u^2\Phi_{pq}\Phi_{lk}\,dxdy \\ &\approx \sum_{e=0}^{N}\sum_{f=0}^{N} u^2(x_e, y_f)\Phi_{pq}(x_e, y_f)\Phi_{lk}(x_e, y_f)w_{ef}\end{aligned} \tag{45}$$

Step4: Solve the discrete model problem by interpolating coefficients Legendre-Galerkin spectral methods

Since $V(x) = x^2 + y^2$ satisfies $\lim_{|x|\to+\infty} V(x) = +\infty$ the solution $\phi(x)$ has an exponential decay property, □ ^{2} can be truncated to the bounded region $\Omega = \left[-L, L\right]^2$ and the chi-squared Dirichlet boundary condition is imposed. The model problem is then discretized by the Legendre-Galerkin spectral method with interpolating coefficients to obtain a system of nonlinear algebraic equations

$$F(u,\mu) = \begin{pmatrix} Ku + \beta Mf(u) - \mu Mu \\ L^2u^T Mu - 1\end{pmatrix} = 0 \tag{46}$$

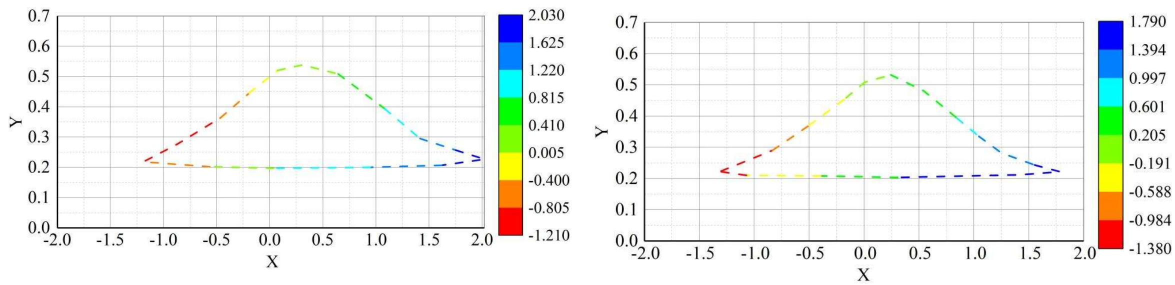
and the corresponding Jacobi:

$$J = \begin{pmatrix} K + \beta MD_f(u) - \mu M & -Mu \\ 2L^2u^T M & 0\end{pmatrix} \tag{47}$$

with $u^0 = \left(U_{11}^0, U_{21}^0, \cdots, U_{N-1,1}^0, U_{12}^0, \cdots, U_{N-1,2}^0, \cdots, U_{1,N-1}^0, \cdots, U_{N-1,N-1}^0\right)^T$, $\mu^0 = \mu_n$ as initial values. The generalized extension method is used to solve this system of nonlinear algebraic equations. where $U_{jk}^0 = \sum_{p,q=0}^{n-1} a_{pq}^0\Phi_{pq}\left(L\xi_j, L\xi_k\right)$, $\left(a^0, \mu_n\right)$ are obtained from Step3.

Numerical results are given below for the two-dimensional harmonic resonator potential well case using convex functions to construct eigenpairs of initial values. Let the initial parameters be $\Omega = (-2,2)^2$, the spectral method discretized by taking $N = 34$, the termination condition of the generalized extension method $\grave{o} = 1.0\times10^{-10}$, the iterative control accuracy $\grave{o} = 1.0\times10^{-8}$, and the number of dissectors problems of the generalized extension method is 4.

(1) Single-feature case: The unique single-feature root $\lambda_{00} = 2$, with $u^0 = a_{00}\Phi_{00}$ and $\mu^0 = 2 + 0.1592\beta$ as initial value. Taking $\beta = -1$, the extended step (Step3) uses 2 feature bases. Figure 2 shows the result of solving the single feature case.

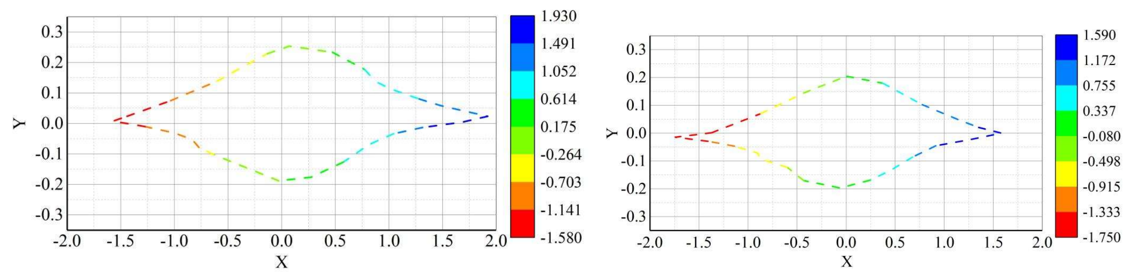


(a) Characteristic function $u^0 = \Phi_{00}(a)$, $\mu = 1.8408$ (b) Corresponding solutions $u_{00}(b)$, $\mu = 1.8280$

Figure 2: Characteristic function and corresponding solutions

(2) 2-fold eigencase: consider the 2-fold eigenvalue $\lambda_{10} = \lambda_{01} = 4$, with the initial value taken as $u^0 = a_{10}\Phi_{10} + a_{01}\Phi_{01}$, and $\mu^0 = 4 + 0.1195\beta$. Taking $\beta = 1$, 8 feature bases are used for the extension step. Figure 3 shows the result of solving the 2-feature case.

Combining Figures 2 and 3, it can be seen that the graphs of the eigenfunction and its initial value solution are very similar. When β is consistent, the number and shape of solutions are consistent. However, the solution peaks when $\beta = -1$ become higher and finer than when $\beta = 1$, which is consistent with the physical facts of the flight trajectory. It shows that accurate trajectory tracking results can be obtained by using the generalized extension method for trajectory tracking solution.



(a) Characteristic function $u^0 = \frac{1}{\sqrt{2}}(\Phi_{10} + \Phi_{01})(a)$, $\mu = 4.1195$ (b) Corresponding solutions $u_{10+01}(b)$, $\mu = 4.1207$

Figure 3: Characteristic function and corresponding solutions

III. C. Stability verification of optimized solution for nonlinear systems

Based on the generalized prolongation method, the gear micro-parameter system and the trajectory tracking system are optimally solved, and it is found that the generalized prolongation method is capable of achieving the optimal solution solving of the gear micro-parameters and the physical prediction of the flight trajectories. Further, the stability of the optimization solution based on the generalized extension method is judged by Ljapunov stability verification.

Definition 1: The solution $u(y,t) = e^{-i\omega t}\left(\hat{h}(y) + i\hat{\ell}(y)\right)$ is systematically stable in V if, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that, when the initial values $\left(\hat{h}(y,0), \hat{\ell}(y,0)\right)^T \in \mathrm{V}$ when there was $\left\|\left(\hat{h}(y,0), \hat{\ell}(y,0)\right)^T - \left(\tilde{h}(y), \tilde{\ell}(y)\right)^T\right\| < \delta$, then

$$\inf_{g \in \mathrm{G}}\left\|\left(\hat{h}(y,t), \hat{\ell}(y,t)\right)^T - T(g)\left(\tilde{h}(y), \hat{\ell}(y)\right)^T\right\| < \varepsilon \tag{48}$$

For this definition, in order to argue for stability, we require that $\hat{h}(y,t)$, $\hat{\ell}(y,t)$ and their fifth-order derivatives are square-productible. We discuss the stability of subharmonic functions with period NT in the space V .

$$\mathrm{V} = H_{per}^5\left(\left[-\frac{NT}{2}, \frac{NT}{2}\right]\right) \times H_{per}^5\left(\left[-\frac{NT}{2}, \frac{NT}{2}\right]\right) \tag{49}$$

To prove nonlinear stability, we first construct a Lyapunov generalized function. For nonlinear systems, this is usually a constant for motion $\mathrm{E}(\tilde{h},\tilde{\ell})$, where $\left(\tilde{\hbar},\tilde{\ell}\right)$ is an unconstrained minimizer of

$$\frac{d\mathrm{E}(\tilde{h},\tilde{\ell})}{dt}=0,\quad \mathrm{E}'(\tilde{h},\tilde{\ell})=0,\quad \langle\varsigma,\mathrm{E}''(\tilde{h},\tilde{\ell})\varsigma\rangle>0,\quad \forall\varsigma\in\mathrm{V},\quad \varsigma\neq 0 \tag{50}$$

Here $\mathrm{E}'\left(\tilde{h},\tilde{\ell}\right)$ denotes the gradient of the generalized function of E. The existence of Liapunov generalized functions leads to formal stability. We need to find the generalized function that satisfies the condition.

With respect to the solution $\left(\tilde{h},\tilde{\ell}\right)$ linearizing the members of the j th NLS cluster yields

$$\mathrm{P}_{t_j}=J\mathrm{M}_j\mathrm{P},\quad \mathrm{P}=\begin{pmatrix}p(y,t)\\ l(y,t)\end{pmatrix} \tag{51}$$

Here M_j is the Hessian matrix of $\hat{I}_j$ at $\left(\tilde{\hbar},\tilde{\ell}\right)^T$. The squared eigenfunction relations as well as the separating variables are given

$$2\Omega_j\mathrm{P}=J\mathrm{M}_j\mathrm{P} \tag{52}$$

Here Ω_j is given by $\varphi(y,t_j)=e^{\Omega_j t_j}v(y)$.

Substituting (52) into (53) yields a relationship between Ω_j and ζ similar to that obtained previously. By direct calculation we have

$$\Omega_j^2(\zeta)=e_j^2(\zeta)\Omega^2(\zeta) \tag{53}$$

where $e_j(\zeta)$ is a polynomial of degree $j-3$ in ζ. Moreover, the choice of parameters $c_{j,n}$, $n>1$, completely controls the roots of $e_k(\zeta)$. For any j, by definition of M_j we have

$$J\mathrm{M}_j\mathrm{P}=2\Omega_j\mathrm{P}\Rightarrow \mathrm{M}_j\mathrm{P}=2\Omega_j J^{-1}\mathrm{P} \tag{54}$$

and

$$K_j=\int_{-\frac{NT}{2}}^{\frac{NT}{2}}\mathrm{P}^*\mathrm{M}_j\mathrm{P}\,dy=2\Omega_j\int_{-\frac{NT}{2}}^{\frac{NT}{2}}\mathrm{P}^*J^{-1}\mathrm{P}\,dy=\frac{\Omega_j}{\Omega}\int_{-\frac{NT}{2}}^{\frac{NT}{2}}\mathrm{P}^*\mathrm{M}\,\mathrm{P}\,dy \tag{55}$$

So we have

$$K_j=\frac{\Omega_j}{\Omega}K_3=e_jK_3 \tag{56}$$

In order to prove the system solution stability, we check the Krein metric K_3 that

$$K_3=\langle\mathrm{P},\mathrm{M}\,\mathrm{P}\rangle=\int_{-\frac{NT}{2}}^{\frac{NT}{2}}\mathrm{P}^*\mathrm{M}\,\mathrm{P}\,dy=4i\Omega\int_{-\frac{NT}{2}}^{\frac{NT}{2}}\left(|\varphi_1|^4-|\varphi_2|^4\right)dy \tag{57}$$

We know that $\varphi_1=-\exp(\Omega t)\gamma(y)B_2$ and $\varphi_2=\exp(\Omega t)\gamma(y)(A_2-\Omega)$. Since Ω is purely imaginary on σ_L, the factor $\exp(\Omega t)$ has no effect. Next we compute the value of $\gamma(y)$. From eqrefd-11 we know that

$$\begin{aligned}\gamma(y)&=\gamma_0\exp\left(-\int\frac{B_{2y}+i\zeta B_2+\psi(A_2-\Omega)}{B_2}dy\right)\\&=\frac{\gamma_0}{B_2}\exp\left(-\int\frac{\psi(A_2-\Omega)}{B_2}dy\right)\exp\left(ireal\right)\end{aligned} \tag{58}$$

where 'real' denotes a real quantity. Thus, the obtained exponent as well as the constant γ_0 has no effect on the modulus of $|\gamma(y)|$, which we equate to 1. For the product function we have

$$
\begin{aligned}
&-\frac{\psi\left(A_2-\Omega\right)}{B_2}=\frac{\psi C_2}{A_2+\Omega} \\
&=\frac{-\beta\psi\psi_{yy}^*-(2i\zeta\beta+i\alpha)\psi\psi_y^*+(4\beta\zeta^2+2\alpha\zeta-V)|\psi|^2+2\beta|\psi|^4}{\frac{i\omega}{2}+i\zeta(V-2\alpha\zeta-4\beta\zeta^2)-i(2\beta\zeta+\alpha)|\psi|^2+\beta(\psi_y\psi^*-\psi\psi_y^*)+\Omega} \\
&=\frac{\beta(\tilde{h}\tilde{\ell}_{yy}-\tilde{\ell}\tilde{h}_{yy})-(\alpha+2\beta\zeta)(\tilde{h}\tilde{h}_y+\tilde{\ell}\tilde{\ell}_y)}{\frac{\omega}{2}+(V-2\alpha\zeta-4\beta\zeta^2)\zeta-(\alpha+2\beta\zeta)(\tilde{h}^2+\tilde{\ell}^2)+2\beta(\tilde{h}\tilde{\ell}_y-\tilde{h}_y\tilde{\ell})-i\Omega} \\
&+imag \\
&=\frac{1}{2}\Big(\ln\Big|-(\alpha+2\beta\zeta)(\tilde{h}^2+\tilde{\ell}^2)+2\beta(\tilde{h}\tilde{\ell}_y-\tilde{h}_y\tilde{\ell}) \\
&+(V-2\alpha\zeta-4\beta\zeta^2)\zeta+\frac{\omega}{2}-i\Omega\Big|\Big)_y+imag
\end{aligned}
\tag{59}
$$

Here 'imag' denotes a purely imaginary quantity that has no effect on the final result of $|\gamma(y)|$. Noting that $A_2+\Omega$ is imaginary, then the absolute value in (59) is real. Thus we get

$$
|\gamma(y)|^2=\frac{|A_2+\Omega|^2}{|B_2|^2}=\frac{1}{|A_2-\Omega|}
\tag{60}
$$

Further we have

$$
|\varphi_1|^4=|\gamma(y)|^4|B_2|^4=|A_2+\Omega|^2,\ |\varphi_2|^4=|\gamma(y)|^4|A_2-\Omega|^4=|A_2-\Omega|^2
\tag{61}
$$

So:

$$
\begin{aligned}
K_3&=4i\Omega\int_{-\frac{NT}{2}}^{\frac{NT}{2}}(|\varphi_1|^4-|\varphi_2|^4)dy=-16i\Omega^2\int_{-\frac{NT}{2}}^{\frac{NT}{2}}A_2dy \\
&=-\frac{64N\Omega^2K(k)(\alpha+2\beta(\zeta-\rho))}{m}\left(m^2\left(k^2+\frac{E(k)}{K(k)}\right)-(\rho+2\zeta)^2\right)
\end{aligned}
\tag{62}
$$

where $E(k)$ is the second class of complete integrals:

$$
E(k)=\int_0^{\pi/2}\sqrt{1-k^2\sin^2 x}dx
\tag{63}
$$

Therefore, there exist two cases $\Omega=0$ or $P(\zeta)=m^2\left(k^2+\frac{E(k)}{K(k)}\right)-(\rho+2\zeta)^2=0$ such that $K_3=0$, and thus we have the following lemma.

Lemma 1: The sign of K_3 changes at $\zeta=\zeta_\pm$, where

$$
\zeta_\pm=\frac{\pm\sqrt{m^2\left(k^2+\frac{E(k)}{K(k)}\right)}-\rho}{2}
\tag{64}
$$

and $\zeta_\pm$ does not fall in σ_L.

Proof: since $K(k)>E(k)$ and $E(k)>K(k)\sqrt{(1-k^2)}$, we have $1-k<\frac{E(k)}{K(k)}<1+k$. It is easy to know that $\zeta_a<\zeta_-<\zeta_b$ and $\zeta_c<\zeta_+<\zeta_d$, and from the previous analysis we know that the intervals (ζ_a,ζ_b) and (ζ_c,ζ_d) are not in the Lax spectrum, and therefore $\zeta_\pm$ does not belong to σ_L.

Based on Lemma 1, we get $K_3=0$ only if ζ is in σ_L such that $\Omega^2=0$. K_3 will have a fixed sign in different parts of σ_L. Since $\hat{H}$ is not a proper Lyapunov generalization, we need to use higher order conserved quantities to generate a new Lyapunov generalization. We will verify the other K_j by considering $K_6 = e_6 K_3$. In order to compute K_6, we utilize Lax for the

$$\hat{\Phi}_{t_6} = \left(U_6 + \sum_{n=0}^{5} c_{6,n} U_n \right) \Phi \tag{65}$$

The sixth linearized NLS equation can be expressed as

$$\frac{\partial}{\partial_{t_6}} \begin{pmatrix} \tilde{h} \\ \tilde{\ell} \end{pmatrix} = J \left(I_6' + \sum_{n=0}^{5} c_{6,n} I_n' \right) = 0 \tag{66}$$

A direct calculation gives

$$\begin{aligned} \Omega_6^2 &= -\frac{1}{4}(2\zeta + \rho + m(k+1))(2\zeta + \rho + m(k-1)) \\ &(2\zeta + \rho - m(k+1))(2\zeta + \rho - m(k-1))F^2(\zeta) \end{aligned} \tag{67}$$

Among them:

$$\begin{aligned} F(\zeta) = {} & m^4(k^4 + 4k^2 + 1) + m^2(k^2 + 1)(10\rho^2 - 4(2\zeta + c_{6,5})\rho \\ & +4\zeta^2 + 2c_{6,5}\zeta - c_{6,4}) + 5\rho^4 - 4(c_{6,5} + 2\zeta)\rho^3 \\ & +(6c_{6,5}\zeta + 12\zeta^2 - 3c_{6,4})\rho^2 \\ & -(16\zeta^3 + 8c_{6,5}\zeta^2 - 4c_{6,4}\zeta - 2c_{6,3})\rho + 16\zeta^4 \\ & +8c_{6,5}\zeta^3 - 4c_{6,4}\zeta^2 - 2c_{6,3}\zeta + c_{6,2} \end{aligned} \tag{68}$$

From (56), we know that

$$\Omega_6^2 = e_6^2 \Omega^2 \tag{69}$$

Thus (56) implies that the choice of the constants $c_{6,2}$, $c_{6,3}$, $c_{6,4}$ and $c_{6,5}$ will determine the sign of K_6. Indeed, $K_6 = e_6 K_3$, where e_6 is a polynomial of degree 3 with respect to ζ. Because we can control the roots of $F(\zeta)$, we pick the appropriate $c_{6,2}$, $c_{6,3}$, $c_{6,4}$ and $c_{6,5}$ such that when the term in K_3 that contains ζ changes sign, $e_6(\zeta)$ changes sign as well. This can be done because the term containing ζ in K_3 is a quadratic polynomial in ζ, which gives K_6 a definite sign over the whole Lax spectrum. Thus we have the following lemma.

Lemma 2: For any $\zeta \in \sigma_L$, the Krein indicator $K_6 > 0$ is satisfied if and only if $c_{6,2}$, $c_{6,3}$, $c_{6,4}$ and $c_{6,5}$ satisfy

$$F(\zeta_{\pm}) = 0, \quad F(\zeta_1) = 0, \quad F_{\zeta}(\zeta_1) = 0 \tag{70}$$

We now know that $\hat{I}_6$ is a Liapunov generalized function with respect to the steady state solution. Thus as long as the solution is spectrally stable with respect to the subharmonic perturbation, then it is formally stable in V. Since symmetric infinitesimal generators correspond to those values of ζ that make $\Omega^2(\zeta) = 0$, the kernel of the generalized function $\hat{I}_{6''}$ consists of the symmetric infinitesimal generators that solve for $\left(\tilde{h}, \tilde{\ell}\right)$. As mentioned earlier, $\zeta_{\pm}$ is not in σ_L. Thus $K_6 = 0$ can only be obtained if $\Omega = 0$ holds with respect to $\zeta \in \sigma_L$. Thus we prove the theorem: (System Stability) For subharmonic perturbations in V, the solution of the generalized extended approximation is stable.

IV. Conclusion

In this paper, we propose a nonlinear system optimization framework that integrates the generalized extension method and Lyapunov stability. In the gear parameter optimization system, the generalized extension method solution error is only between [-0.12,0.05], and most of them are between 0.00-0.02, with more concentrated error distribution. In trajectory tracking optimization, the graphs of the eigenfunction and its initial value solution are very similar, and the physical consistency error is smaller. The Liapunov stability analysis verifies that the method in this paper is spectrally stable under subharmonic perturbations. The computational speed of the high-dimensional spatial projection operator can be enhanced in the future to expand the possibility of applying the generalized extension method in real-time optimization systems.

References

[1] Branislav, I., Haifeng, M., & Dijana, M. (2020). A survey of gradient methods for solving nonlinear optimization. Electronic research archive, 28(4), 1573-1624.

[2] Vagaská, A., & Gombár, M. (2021). Mathematical optimization and application of nonlinear programming. Algorithms as a basis of modern applied mathematics, 461-486.

[3] Mikishanina, E. (2023). Nonlinear Optimization Logistic Model in the Problem of Cargo Transportation. Transportation Research Procedia, 68, 133-137.

[4] Abbas, G., Gu, J., Farooq, U., Asad, M. U., & El-Hawary, M. (2017). Solution of an economic dispatch problem through particle swarm optimization: a detailed survey-part I. IEEE Access, 5, 15105-15141.

[5] Arslan, C., Karasakal, O., & Kırca, Ö. (2024). Naval Air Defense Planning problem: A novel formulation and heuristics. Naval Research Logistics (NRL), 71(7), 895-919.

[6] Curtis, F. E., Wachter, A., & Zavala, V. M. (2018). A sequential algorithm for solving nonlinear optimization problems with chance constraints. SIAM Journal on Optimization, 28(1), 930-958.

[7] Liu, Z. Z., & Wang, Y. (2019). Handling constrained multiobjective optimization problems with constraints in both the decision and objective spaces. IEEE Transactions on Evolutionary Computation, 23(5), 870-884.

[8] Huang, X., & Cui, B. (2019). A neural dynamic system for solving convex nonlinear optimization problems with hybrid constraints. Neural Computing and Applications, 31(10), 6027-6038.

[9] Lu, H. C., Tseng, H. Y., & Lin, S. W. (2023). Double-track particle swarm optimizer for nonlinear constrained optimization problems. Information Sciences, 622, 587-628.

[10] Abdelhalim, A., Nakata, K., El-Alem, M., & Eltawil, A. (2018). Guided particle swarm optimization method to solve general nonlinear optimization problems. Engineering Optimization, 50(4), 568-583.

[11] Meng, F., Shen, X., & Karimi, H. R. (2022). Emerging methodologies in stability and optimization problems of learning-based nonlinear model predictive control: a survey. International Journal of Circuit Theory and Applications, 50(11), 4146-4170.

[12] Ajagekar, A., Humble, T., & You, F. (2020). Quantum computing based hybrid solution strategies for large-scale discrete-continuous optimization problems. Computers & Chemical Engineering, 132, 106630.

[13] Hayyolalam, V., & Kazem, A. A. P. (2020). Black widow optimization algorithm: a novel meta-heuristic approach for solving engineering optimization problems. Engineering Applications of Artificial Intelligence, 87, 103249.

[14] Garg, V., Deep, K., & Bansal, S. (2023). Improved Teaching Learning Algorithm with Laplacian operator for solving nonlinear engineering optimization problems. Engineering Applications of Artificial Intelligence, 124, 106549.