International Journal for Housing Science and Its Applications
Publish August 10, 2025. Volume 46, Issue 4 Pages 8008-8017

https://doi.org/10.70517/ijhsa464686

Study of quasi-periodic solutions of the unbounded non-homogeneous kdv-mkdv equation

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Abstract This paper systematically studies the qualitative properties of solutions and the construction of quasi-periodic solutions for the non-homogeneous KdV-mKdV equations with unbounded boundary conditions. Based on prior estimation theory and the definition of uniform fractional derivatives, continuous dependence and boundedness estimates for the solutions of the equations are established. By transforming the equations into ordinary differential equations via a traveling wave transformation, the waveform stability of the traveling wave solutions is revealed. Numerical simulation methods are used to verify the long-term conservation of soliton solutions, and the homogeneous equilibrium method and Maple calculations are employed to construct the analytical form of quasi-periodic solutions. The results indicate that under unbounded non-homogeneous conditions, the behavior of the solutions to Equation $u_t + 2\alpha u u_x - 3\beta u^2 u_x + \varepsilon u_{xxx} = 0$ is significantly influenced by the sign of the nonlinear term and the parameter matching relationship, and the quasi-periodic solutions exhibit rich dynamical characteristics ranging from localized freak waves to asymptotically periodic waves.

Index Terms KdV-mKdV equations, unbounded conditions, homogeneous equilibrium method, quasi-periodic solutions, numerical simulation

I. Introduction

With the rapid development of science and technology, the focus of scientific research has gradually shifted from linear to nonlinear phenomena [1]. Nonlinear phenomena (such as solitons, chaos, and fractals) have been discovered in many fields, and the corresponding mathematical models—nonlinear equations—have emerged [2], [3]. The KdV-mKdV equations are the earliest and most representative nonlinear equations, representing the coupled form of the KdV and mKdV equations [4], [5].

The KdV equation is an important mathematical model for describing physical phenomena such as water waves and plasma waves. The KdV equation possesses numerous significant properties and analytical solutions [6], [7]. It is an integrable system, meaning it has an infinite number of conservation laws, enabling analytical methods to solve the equation [8], [9]. The most famous analytical solution is the solitary wave solution, also known as the KdV soliton [10]. A soliton is a special type of wave that maintains its shape and velocity unchanged during propagation, exhibiting highly stable properties [11], [12]. The study of the KdV equation is not limited to mathematical physics but also has broad practical applications [13]. For example, in hydrodynamics, the KdV equation can be used to describe tidal wave phenomena in rivers and oceans, helping us better understand the movement patterns of oceans and rivers. In nonlinear optics, the KdV equation can be used to describe the propagation of light waves in optical fibers, with important applications in fields such as optical communication and optical sensing [14]-[17]. The mKdV equation is a modified version of the KdV equation. The KdV-mKdV equation has significant application prospects in mathematics, physics, and engineering. In recent years, research on the exact solutions of this equation has been increasing [18]-[20].

This paper first defines consistent fractional derivatives and derives their operational rules. By combining Gronwall's inequality, it establishes a priori estimates for the solutions of the equation. Using phase plane qualitative analysis methods, it systematically discusses the changes in waveforms under the combined effects of nonlinear terms, dissipative terms, and dispersion terms. Through numerical simulations involving the selection of spatial and temporal step sizes and the setting of multi-layer initial conditions, it verifies the waveform conservation of solitary wave solutions. Using the homogeneous equilibrium method combined with Maple calculations, the analytical form and evolutionary characteristics of quasi-periodic solutions are explored.

II. Solving non-homogeneous kdv-mkdv equations with unbounded conditions

II. A. Definition of the KdV-Burgers equation

The KdV-Burgers equation is abstracted from many physical phenomena (such as turbulence problems, liquid flow containing bubbles, liquid flow in elastic pipes, etc.) and has broad physical and practical significance. Therefore, research on this equation has certain theoretical and practical significance. Two-dimensional generalized non-homogeneous KdV-Burgers mixed-type equation:

$$\{u_t + [f(u)]_x + \alpha u_{xx} + \beta u_{xxx}\}_x + \delta u_{yy} = g \quad t \geq 0, x, y \in R \tag{1}$$

where $[f(u)]_x$ is the nonlinear term, αu_{xx} is the dissipative term, α is the dissipation coefficient, $\beta\mu_{xxx}$ is the dispersion term, and β is the dispersion coefficient. g is a function of x, y, t and is the forced term, α, β, δ are nonzero constants.

This paper further discusses the continuous dependence of the solution, the boundedness of the solution, and the estimation formula. The discussion of problem (1) is ultimately equivalent to the discussion of the inverse periodic boundary value problem:

$$\begin{cases} -\phi''(z) + c\phi'(z) + d\phi(z) + F[\phi(z)] = G(Z) & 0 \leq z \leq T \\ \phi^{(i)}(T) = -\phi^{i}(0) & i = 0,1 \end{cases} \tag{2}$$

Among them, $F[\phi(z)] = ef[\phi(z)]$.

Therefore, we will discuss the continuous dependence of the solution, the boundedness of the solution, and the estimation formula for question (2) below.

II. B. Prior estimation

Lemma 1 If $u_0 \in L_2(\Omega), g(x) \in L_2(\Omega), \Omega = (0, \infty)$, then the following estimate holds for the global smooth solution to the problem:

$$\| u \|^2 \leq \frac{1}{\beta^2}(1 - e^{-\beta}) \| g(x) \|^2 + e^{-\beta} \| u_0 \|^2 \tag{3}$$

Proof: Take the inner product with u, that is,

$$(u, u_t + f(u)_x + u_{xxx} - \alpha u_{xx} + \beta u) = (g(x), u) \tag{4}$$

Among them

$$(u, u_t) = \frac{1}{2}\frac{d}{dt}\| u \|^2$$

$$(u, u_{xxx}) = u_{xx}u \big|_0^{+\infty} - \int_0^{+\infty} u_x u_{xx} dx = -\frac{1}{2}u_x^2 \big|_0^{+\infty} = \frac{1}{2}u_x^2(0) \geq 0$$

$$(u, -\alpha u_{xx}) = -\alpha(uu_x \big|_0^{+\infty} - \int_0^{+\infty} u_x^2 dx) = \alpha \| u_x \|^2, (u, \beta u) = \beta \| u \|^2 \tag{5}$$

$$(u, f(u)_x) = uf(u) \big|_0^{+\infty} - \int_0^{+\infty} f(u)u_x dx = -\int_0^{+\infty} F(u)_x dx = -F(u) \big|_0^{+\infty} = 0$$

$$(u, g(x)) \leq \frac{\beta}{2}\| u \|^2 + \frac{1}{2\beta}\| g(x) \|^2$$

From (4), we obtain

$$\frac{1}{2}\frac{d}{dt}\| u \|^2 + \alpha \| u_x \|^2 + \frac{\beta}{2}\| u \|^2 \leq \frac{1}{2\beta}\beta(x) \|^2 \tag{6}$$

Then, according to Gronwall's inequality, from (6), we obtain (3). Lemma 1 is proved.

Lemma 2 Under the conditions of Lemma 1, assume that

(1) $f(u) \in C^2, | f(u) | \leq A | u |^{5-\delta}, \delta > 0, A > 0$;

(2) $u_{0x} \in L_2(\Omega)$.

Then, for the global smooth solution to the problem, we have the following estimate

$$\| u_x \|^2 \leq 2e^{-2\beta}\Psi(u_0) + \frac{1}{\beta}(1 - e^{-2\beta})\left[\frac{2}{\alpha}\| g(x) \|^2 + C_5(E_0)\right] + 2C_6(E_0) \tag{7}$$

Among them, the functions $C_5(\cdot), C_6(\cdot)$ are only related to $\| u(\cdot, t) \|$, and

$$\Psi(u) = \| u_x \|^2 - 2\int F(u)dx, F(u) = \int_0^u f(s)ds \tag{8}$$

Proof: Take the inner product with u_{xx}, that is,

$$(u_{xx}, u_t + f(u)_x + u_{xxx} - \alpha u_{xx} + \beta u) = (g(x), u_{xx}) \tag{9}$$

Among them

$$\begin{aligned}
&(u_{xx}, u_t) = u_x u_t |_0^{+\infty} - \int_0^{+\infty} u_x u_{xt} dx = -\frac{1}{2}\frac{d}{dt} \| u_x \|^2 \\
&(u_{xx}, u_{xxx}) = \frac{1}{2} u_{xx}^2 |_0^{+\infty} = -\frac{1}{2} u_{xx}^2(0,t) = 0, (u_{xx}, -au_{xx}) = -\alpha \| u_{xx} \|^2 \\
&(u_{xx}, \beta u) = \beta u_x u |_0^{+\infty} - \int_0^{+\infty} u_x u_x dx = -\beta \| u_x \|^2 \\
&(u_{xx}, g(x)) \le \| u_{xx} \| \| g(x) \| \le \frac{\alpha}{8} \| u_{xx} \|^2 + \frac{2}{\alpha} \| g(x) \|^2
\end{aligned} \tag{10}$$

And

$$| 2\int F(u) dx | \le A \| u \|^{6-\delta} \le \frac{\alpha}{4\beta} \| u_{xx} \|^2 + C_4(E_0) \tag{11}$$

Then, from (9), we obtain

$$\frac{d}{dt}(\| u_x \|^2 - 2\int F(u) dx) + 2\beta \| u_x \|^2 + \frac{\alpha}{2} \| u_{xx} \|^2 \le \frac{2}{\alpha} \| g(x) \|^2 + C_5(E_0) \tag{12}$$

$$\frac{d}{dt}(\| u_x \|^2 - 2\int F(u) dx) + 2\beta(\| u_x \|^2 - 2\int F(u) dx) \le \frac{2}{\alpha} \| g(x)^2 \| + C_5(E_0) \tag{13}$$

order

$$\Psi(u) = \| u_x \|^2 - 2\int F(u) dx \tag{14}$$

Obviously $\Psi(u) \ge \frac{1}{2} \| u_x \|^2 - C_6(E_0)$.

Therefore, there is

$$\frac{d}{dt}\Psi(u) + 2\beta\Psi(u) \le \frac{2}{\alpha} + C_5(E_0) \tag{15}$$

$\Psi(u) \le e^{-2\beta}\Psi(u_0) + \frac{1}{2\beta}(1 - e^{-2\beta})\left[\frac{2}{\alpha} \| g(x) \|^2 + C_5(E_0)\right]$. Then, according to Gronwall's inequality, i.e., (13) implies (7).

Lemma 2 is proven.

II. C.Consistent fractional derivatives

Definition 1 The α th-order fractional derivative of a function $f:[0,\infty) \to R$ is defined as

$$T_\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}$$
$$t > 0, \alpha \in (0,1) \tag{16}$$

Let f^α denote this. If $T_\alpha(f)$ exists, then define $f^\alpha(0) = \lim_{t \to 0^+} f^\alpha(t)$.

Lemma 3 Let $\alpha \in (0,1]$ and f, g be α -times differentiable. Then the following conclusions hold:

(1) $T_\alpha(af + bg) = aT_\alpha(f) + bT_\alpha(g), \forall a, b \in R$;

(2) $T_\alpha(t^p) = pt^{p-\alpha}, \forall p \in R$;

(3) $T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$;

(4) $T_\alpha(\frac{f}{g}) = \frac{gT_\alpha(f) - fT_\alpha(g)}{g^2}$;

(5) $T_\alpha(f)(t) = t^{1-\alpha} f'(t)$.

Lemma 4 Let $f, g:(0,\infty) \to R$ be a α nd-order differentiable function, where $\alpha \in (0,1)$, and let g be differentiable at , and f be differentiable at $g(t)$, then

$$T_\alpha(f \circ g)(t) = t^{1-\alpha} g(t)^{\alpha-1} g(t) T_\alpha(f)(g(t)) \tag{17}$$

Based on the above definition and lemma, we can see that the main difference between fractional-order differentiation and integer-order differentiation is the power of the derivative. The fractional order is $1-\alpha$ power, while the integer order is $\alpha - 1$ power. Therefore, when solving the time fractional-order KdV equation, we need to pay attention to the power of time t.

III. Numerical simulation of quasi-periodic solutions of non-homogeneous KdV-mKdV equations with unbounded conditions

III. A. Traveling wave solution

The simplest nonlinear wave equation

$$\frac{\partial u}{\partial t}+\alpha u\frac{\partial u}{\partial x}=0 \tag{18}$$

where α is a non-zero constant. Due to the presence of nonlinear terms in the equation, the waves exhibit a chasing phenomenon, eventually becoming steeper and forming discontinuous solutions. However, when the equation contains only dissipative or dispersive terms, this phenomenon can be terminated, resulting in stable shock or soliton solutions. Now, we first discuss the general case where the equation simultaneously contains nonlinear, dissipative, and dispersive terms, i.e., we discuss the equation

$$\frac{\partial u}{\partial t}+\alpha u\frac{\partial u}{\partial x}+\beta\frac{\partial^2 u}{\partial x^2}+\gamma\frac{\partial^3 u}{\partial x^3}=0 \tag{19}$$

where α,β and γ are nonzero constants. To find the traveling wave solution of equation (19), let

$$u=U(X),X=x-ct \tag{20}$$

Among them, $C\neq o$ is a constant representing the speed of the traveling wave. Substituting (20) into equation (19) yields

$$\gamma\frac{d^3U}{dX^3}+\beta\frac{d^*U}{dX^2}+\alpha U\frac{dU}{dX}-C\frac{dU}{dX}=0 \tag{21}$$

Integration—Once, for the sake of simplicity in the following discussion, appropriate initial conditions are selected so that the integration constant is zero, and equation (21) becomes

$$\gamma\frac{d^2U}{dX^2}+\beta\frac{dU}{dX}+\frac{\alpha}{2}U^2-CU=0 \tag{22}$$

Equation (22) is a nonlinear second-order ordinary differential equation that is difficult to solve using elementary methods. Below, we discuss it using qualitative methods. Its equivalent system of equations is

$$\frac{dU}{dX}=V,\quad \frac{dV}{dX}=\frac{C}{\gamma}U-\frac{\alpha}{2\gamma}U^2-\frac{\beta}{\gamma}V \tag{23}$$

Shilling $a>0,C>0$. Reference transformation

$$\zeta=\frac{\alpha}{2C}U \tag{24}$$

Equation (22) simplifies to

$$\frac{\gamma}{C}\frac{d^2\xi}{dX^2}+\frac{\beta}{C}\frac{d\xi}{dX}+\zeta^2-\zeta=0 \tag{25}$$

Its equivalent equation is

$$\frac{d\zeta}{dX}=\eta,\frac{d\eta}{dX}=\frac{C}{\gamma}(\zeta-\xi^2)-\frac{\beta}{\gamma}\eta \tag{26}$$

(1) When $r>0$, let the transformation be $X=\sqrt{\frac{\gamma}{C}}\xi$, and

Equation (25) is further simplified to

$$\frac{d^2\xi}{d\xi^2}+\frac{\beta}{\sqrt{\gamma C}}\frac{d\xi}{d\xi}+\xi^2-\xi=0 \tag{27}$$

Its equivalent system of equations is

$$\frac{d\xi}{d\xi}=\eta,\quad \frac{d\eta}{d\xi}=\xi-\xi^2-\lambda\eta \tag{28}$$

where $\lambda=\frac{\beta}{\sqrt{\gamma C}}$, λ and β have the same sign.

In the phase plane (ζ,η), the singular points of the system of equations (28) are (0, 0) and (1, 0). In the non-critical case, the stability of the zero solutions of the system of equations (1, 11) can be used in the relevant linear approximate system of equations.

$$\frac{d\xi}{d\xi}=\eta,\quad \frac{d\eta}{d\xi}=\zeta-\lambda\eta \tag{29}$$

Since the roots of the characteristic equation of the system of equations (29) are

$$K^2+\lambda K-1=0 \tag{30}$$

Since $\lambda^2+4>0, \lambda\neq 0$ (i.e., $\beta\neq 0$), $K_{1\cdot 2}$ are two roots of opposite signs, at this point, the singular point (0, 0) of the linear system (29) is a saddle point, and the zero solution is unstable. Therefore, the singular point (0, 0) of the related nonlinear system (29) is also a saddle point, and the zero solution is unstable.

To examine the stability at the singular point (1, 0), we perform a transformation.

$$\zeta^*=\zeta-1, \eta^*=\eta \tag{31}$$

Equation group (29) becomes

$$\frac{d\xi^*}{d\xi}=\eta^*, \frac{d\eta^*}{d\xi}=-\xi^*-\xi^* z-\lambda\eta^* \tag{32}$$

Its corresponding system of linear approximation equations is

$$\frac{d\zeta^*}{d\xi}=\eta^*,\quad \frac{d\eta^*}{d\xi}=-\zeta^*-\lambda\eta^* \tag{33}$$

The roots of its characteristic equation are

$$K_{1\cdot 2}=\frac{-\lambda\pm\sqrt{\lambda^2-4}}{2} \tag{34}$$

When $\lambda^*-4\geq 0$ (i.e., $\beta^*\geq 4\gamma C$), $\lambda>0$ (i.e., $\beta>0$), the characteristic roots are two distinct negative real roots, two equal negative real roots, and two complex conjugate roots with negative real parts, respectively. Then, the singularity (0, 0) of the system of equations (33) is a stable node, a stable degenerate node, and a stable focus, respectively. The zero solution is asymptotically stable, so the singular point (0, 0) of the related nonlinear system of equations (32) is a node type or focus type, and the zero solution is asymptotically stable. That is, the singular point (1, 0) of the system of equations (29) is a stable node type or focus type, and the solution (1, 0) is asymptotically stable. Using a completely analogous method to discuss the case when $\gamma>0$ and $\beta<0$, we find that the singular point (1, 0) of the system of equations (29) is an unstable node or focus, and the solution (1, 0) is unstable.

(2) When $\gamma<0$, let the transformation be

$$X=\sqrt{-\frac{\gamma}{C}\xi} \tag{35}$$

Equation (25) becomes

$$\frac{d^2\xi}{d\xi^2}-\frac{\beta}{\sqrt{-\gamma c}}\quad\frac{d\zeta}{d\xi}-\zeta^2+\zeta=0 \tag{36}$$

Its equivalent system of equations is

$$\frac{d\zeta}{d\xi}=\eta,\quad \frac{d\eta}{d\xi}=-\zeta+\zeta^2+\mu\eta \tag{37}$$

where $\mu=\frac{\beta}{\sqrt{-\gamma c}}$, and μ and β have the same sign.

In the complex plane (ζ,η), the singular points of the system of equations (37) are still (0, 0) and (1, 0). Using the same steps as those discussed earlier for $\gamma>0$, the corresponding results can be obtained by analogy. For ease of comparison, the results of the discussion on system (26) are shown in Table 1. The characteristic abbreviations in Table 1 are included solely for the sake of simplicity. When α and C are both greater than 0, different signs for γ and β produce different waveforms W1~W4.

Table 1: In the case where α > 0 and C > 0

γ	+	+	-	-
β	+	-	+	-
Singularity (0,0)	Saddle point type	Saddle point type	Unstable junction,focal type	Stable junction,focal type
Solution (0,0)	Unstable	Unstable	Unstable	Asymptotic stability
Abbreviation character	US	US	UNF	SNF
Singularity (1,0)	Stable junction,focal type	Unstable junction,focal type	Saddle point type	Saddle point type
Solution (1,0)	Asymptotic stability	Unstable	Unstable	Unstable
Abbreviation character	SNF	UNF	US	US
Wave type	W1	W2	W3	W4

In the solution of the equation for traveling waves, the ordinary differential equation encountered contains four constants α,β,γ, and C. The different signs of these constants will affect the direction, position, type, and stability of the waves, making the situation quite complex. Based on the characteristics of the qualitative analysis results above, it can be determined that the waveforms determined by equation (22) are non-periodic decaying (or divergent) motions and decaying (or divergent) vibrations. The approximate waveforms W_1 to W_4 that produce vibration conditions are shown in Figure 1, all of which are right-propagating waves. It can be concluded that the occurrence of vibration depends on whether the characteristic roots are a pair of conjugate complex numbers, i.e., whether the following conditions are satisfied: when γ, C have the same sign, $\beta^2 < 4\gamma C$; when γ, C have different signs, $\beta^2 < -4\gamma C$.

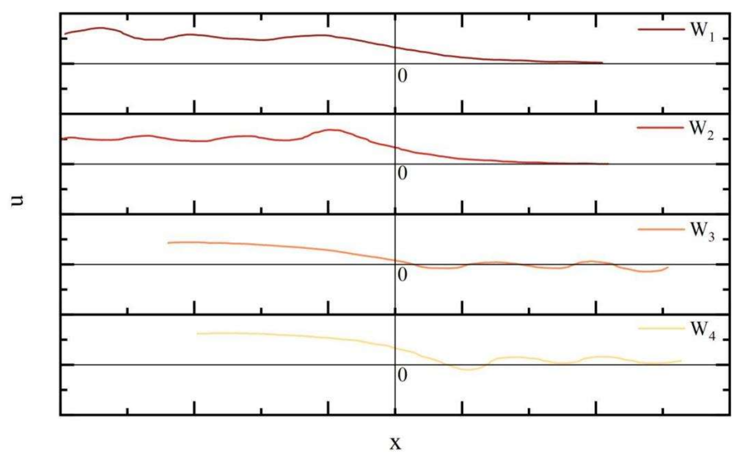

Figure 1: Wave patterns W1 right - W4 right

III. B. Soliton solutions

Consider the following initial value problem for the KdV-mKdV coupled equations used to describe nonlinear lattice propagation wave processes.

$$\begin{cases} \partial u + \partial_{xxu} + (6+6u)u\partial u = 0 \\ u(0,x) = \tanh^2 x, \quad -\infty <_x < \infty \end{cases} \tag{38}$$

There is a solitary wave solution in $x \in (-\infty, \infty)$.

$$u(t,x) = -\frac{1}{2}\left\{1-\tanh\left[\frac{1}{2}(x-t)-\frac{\ln 6}{2}\right]\right\} \tag{39}$$

In this paper, the solitary wave solution is simulated within the range [-20, 10] with a time step of $\Delta t = 0.0005$ and a spatial step of $\Delta x = 0.05$. During the simulation, the first and second layers use the exact solution. The results of simulating the soliton solution within the region $(x,t) \in [-20,10] \times [0,50]$ are shown in Figure 2. The local energy error and local momentum error for $t \in [0,50]$ are shown in Figure 3. As shown in Figure 2, the simulation effectively reproduces the waveform of the original soliton solution while preserving its original waveform. As shown in Figure 3, the simulation not only reproduces the waveform of the original soliton solution and preserves its original waveform but also effectively maintains the local properties of the system over an extended period.

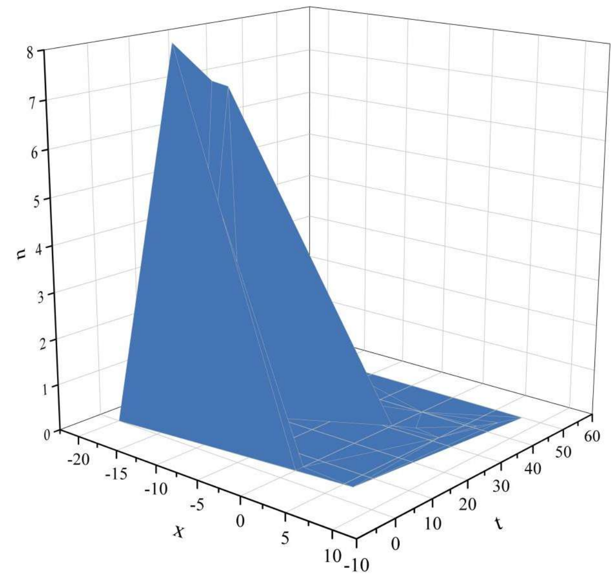

Figure 2: Simulation results of solitary wave solutions

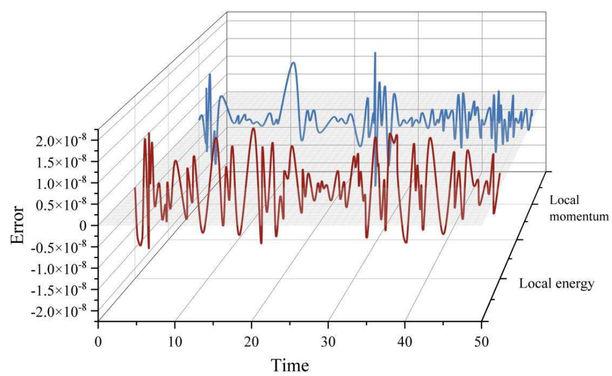

Figure 3: Local energy error and local momentum error

III. C. Proposed periodic solution

When $\gamma = 1$, the generalized KdV-mKdV equation is

$$u_t + 2\alpha u u_x - 3\beta u^2 u_x + \varepsilon u_{xxx} = 0 \tag{40}$$

Let $\xi = x - mt$, then equation (40) becomes

$$-m u_\xi + 2\alpha u u_\xi - 3\beta u^2 u_\xi + \varepsilon u_{\xi\xi\xi} = 0 \tag{41}$$

Furthermore, integrating both sides of equation (41) with respect to ξ yields

$$m u - \alpha u^2 + \beta u^3 - \varepsilon u_{\xi\xi} + \kappa = 0 \tag{42}$$

Among them, κ is the integration constant. For the highest-order derivative term $u_{\xi\xi}$ and the highest-order nonlinear term u^3 in nonlinear equation (42), the homogeneous balance method is used. the final high-order term N in the undetermined expression $u(\xi) = \sum_i = -N^N a_i (d + \frac{G'}{G})^i$ satisfies the equation $N + 2 = 3N$, and the solution yields $N = 1$.

Let $H(\xi) = G' / G$. Assuming that equation (42) has a solution of the following form:

$$u(\xi) = a_1 (d + H(\xi))^{-1} + a_0 + a_1 (d + H(\xi)) \tag{43}$$

Among these, a_0, a_1, a_2, d are undetermined parameters. Substituting equations (3) and (12) into equation (11) and performing calculations using Maple software, the equation can be transformed into a polynomial in terms of $(d+H(\xi))^N (N=0,\pm1,\pm2,\cdots)$. Further setting the coefficients of the different powers of $d+H(\xi)$ to zero yields a system of algebraic equations involving the relevant parameters. Solving these algebraic equations using Maple software yields the following relationships between the parameters.

When $A\neq 0, \beta\varepsilon>0$,

$$\begin{aligned} &a_1=\frac{\mp\sqrt{2}\Phi\varepsilon}{A\sqrt{\beta\varepsilon}}, a_0=\frac{\pm\sqrt{2\beta\varepsilon}(B-2Cd)}{2\beta A}+\frac{\alpha}{3\beta} \\ &a_2=\frac{\pm\sqrt{2\beta\varepsilon}C}{\beta A}, m=\frac{6C\Phi\varepsilon}{A^2}-\frac{\Psi\varepsilon}{2A^2}+\frac{\alpha^2}{3\beta} \\ &\kappa=\frac{\pm2\sqrt{2}C\Phi\varepsilon^2(B-2Cd)}{A^3\sqrt{\beta\varepsilon}}-\frac{\alpha\varepsilon(4C(2\Phi-Cd^2+Bd)-B^2)}{6A^2\beta}-\frac{\alpha^3}{27\beta^2} \end{aligned} \tag{44}$$

In this case, substituting the above parameter relationship into equation (43) yields the following new exact solution expression.

$$\begin{aligned} u_{1i}=&\left(\frac{\mp\sqrt{2}\Phi\varepsilon}{A\sqrt{\beta\varepsilon}}\right)(d+H(\xi))-1+\frac{\pm\sqrt{2\beta\varepsilon}(B-2Cd)}{2\beta A} \\ &+\frac{\alpha}{3\beta}+\frac{\pm\sqrt{2\beta\varepsilon}C}{\beta A}\times(d+H(\xi)),(i=1,2,3,4,5) \end{aligned} \tag{45}$$

Among them, $\xi=x-mt, H(\xi)$ satisfies the equations $\Phi=-Cd^2+Bd+A-E, \Psi=B^2+4C(A-E)$. When the parameters satisfy the corresponding conditions and relationships, the above solution is the new exact solution form.

Using Maple, the solution u_{11} is obtained as shown in Figure 4. From Figure 4, it can be seen that in the short term, the soliton solutions of the generalized KdV-mKdV equation mainly appear in small narrow regions, and sudden changes in strange waves may occur. From the expression of the solution, when the denominator approaches 0, a blow-up phenomenon occurs.

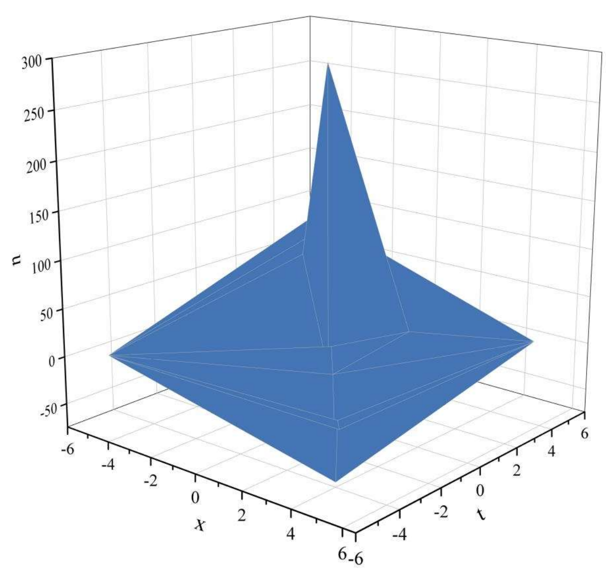

Figure 4: Solution u_{11}

Similarly, when the free parameters are values under different conditions, the specific forms of the corresponding solutions can be obtained. When $B\neq 0, \Omega<0$, take $\varepsilon=\beta=-1, A=2, \alpha=B=C=E=d=C_2=1$, $\Omega=B^2+4E(A-C)=-3<0, C_1=\frac{1}{2}$ is taken. Using Maple, the solution u_{12} is obtained as shown in Figure 5. From Figure 5, it can be seen that within a short period of time, the periodic solution of the generalized KdV-mKdVmKdV appears in a relatively large region. The destructive characteristics may be asymptotic and periodic within a certain period of time.

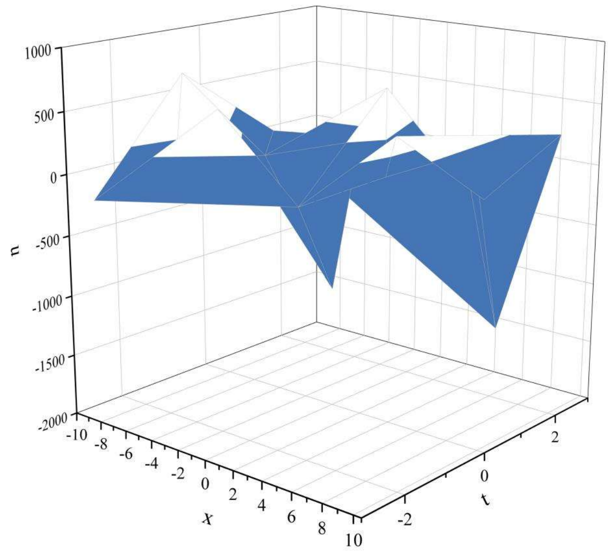

Figure 5: Solution u_{12}

IV. Conclusion

This paper investigates quasi-periodic solutions to the non-homogeneous KdV-mKdV equations with unbounded boundary conditions. The main conclusions are as follows.

(1) Waveform stability of traveling wave solutions: The sign combination of the eigenvalues determines the type of singularity and the asymptotic behavior of the solution. When γ, C have the same sign, $\beta^2 < 4\gamma C$; when γ, C have opposite signs, $\beta^2 < -4\gamma C$.

(2) Numerical verification of soliton solutions: Within the interval [-20, 10], simulations were conducted with a time step of $\Delta t = 0.0005$ and a spatial step of $\Delta x = 0.05$. The numerical simulation results are in good agreement with the analytical solutions, and the local energy error and momentum error remain stable over time, confirming the reliability of the numerical method.

(3) Construction and characteristics of quasi-periodic solutions: At short time scales, the solutions exhibit localized soliton sudden changes. At long time scales, under certain parameter conditions, the solutions exhibit asymptotic periodicity, revealing the complex dynamical responses of nonlinear systems under parameter variations.

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