International Journal for Housing Science and Its Applications
Publish August 10, 2025. Volume 46, Issue 4 Pages 798-813

https://doi.org/10.70517/ijhsa46473

Calculation of critical exponents for noncommutative maximal subgroups of a finite group

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Abstract Non-exchangeable extremely large subgroups have important application value in the study of the nature of groups, which can reveal the intrinsic structure and behavioral patterns of groups. In this paper, a method of calculating the critical index of non-exchangeable extremal subgroups of a finite group is proposed. By introducing the Sylow subgroup of the group and the structural characteristics of the noncommutative extremal subgroup, several theorems and lemmas are proposed to derive the lower bound of the critical exponent. The validity of the method is verified by numerical simulation analysis, and the results show that the number of non-commutative extremal subgroups is significantly affected by the solvability of the group and the informality of the Sylow subgroup in different types of group structures. In the simulation, taking the karate club network as an example, the expectation value of 0.0682 is obtained through 1200 times of random graph simulation calculations, which indicates that the proposed method has high accuracy in practical applications. In addition, for the dolphin group network and the political book classification network, the simulation results also show that the calculation method can effectively reflect the intrinsic relationship and structure of the group. The study shows that the calculation method of the critical exponent of noncommutative extremely large subgroups has potential application value in finite group theory, and can provide an important basis for the classification and structure analysis of groups.

Index Terms Non-commutative extremal subgroup, Critical index, Finite group, Sylow subgroup, Numerical simulation, Group classification

I. Introduction

The concept of group was originally proposed by the famous French mathematician Galois in order to solve the problem of solving equations of fifth and more than fifth order, and the theory of finite groups has long been an important topic of interest for mathematicians [1], [2]. Finite group theory is not limited to the discipline of mathematics but is also an important tool for other disciplines. Utilizing the exchangeability, power-zero, super-solvability, solvability, regularity, pseudo-regularity, S-proposed regularity, and anti-regularity of subgroups to study the structure of finite groups has been a hot topic in the study of finite group theory [3]-[5]. The center is an important characteristic subgroup of the group. Obviously, the smaller the center of the group is, i.e., the fewer elements are exchanged with all the elements in the group, at which time the group can be considered to be weaker in exchange. Conversely, the larger the center of the group, the stronger the exchangeability of the group can be considered, and G is the exchange group when and only when the center of G Z(G)=G [6]-[8].

A group G is said to have a division if G=UHi<G Hi, where Hi∩Hj=1 and i≠j. Each true subgroup here is called a constituent of the group G. This notion was first introduced in 1927 in (J.W. Young), and was mainly studied for the case where G is a commutative group [9]. It is worth noting that as early as 1906, G.A. Miller, although he did not use the terminology introduced by J.W. Young, studied finite exchange groups and pm-ordered non-exchange groups with divisions [10]. The problem of division of groups is of great importance in the study of algebraic problems in general and group theory in particular. In recent years, the study on noncommutative maximal subgroups has accounted for a significant proportion in the study of finite groups, and its critical index calculation presents that the frequency of appearance of noncommutative maximal subgroups is more than half of finite groups up to order 120. Whereas the correlation coefficient of the critical exponent with the finite group is more than 50%, the subgroup lattice search space grows linearly, leading to the complexity of the critical exponent calculation [11]-[13].

The research goal of this paper is to propose a method for calculating the critical index of non-exchangeable extremely large subgroups applicable to general finite groups. The critical index is a measure of the number of non-exchangeable extremal subgroups of the group, which can help us understand the structural characteristics of the group more precisely. We derive the formula for calculating the critical index by analyzing the Sylow subgroup of the group and the basic properties of the noncommutative extremal subgroup. Through a series of numerical simulation experiments, we verify the validity of the proposed method and show its application effect in real networks. Specifically, firstly, we start from the basic theory of finite groups to define and derive the computation method of the critical exponent of non-commutative extremal subgroups. During the construction of the method, we fully consider the solvability of the group, the structure of Sylow subgroups and other basic properties of the group. Second, in order to verify the feasibility of the method, numerical simulations, especially experiments based on random graph models, are used to simulate the behavior of finite groups and analyze the number of non-exchangeable extremely large subgroups. Finally, the simulation of real networks (e.g., karate club network, dolphin group network, etc.) is used to demonstrate the effectiveness of the method in practical problems, further proving its potential value in finite group theory and network analysis.

II. Finite groups whose noncommutative subgroups all have minimal centralizers

Call a finite noncommutative group G a MC group if all noncommutative subgroups of G have minimal centralizers. In particular, if the MC group G is the p group, then G is said to be the MCp group. If the MC group G is a special p group, then call G a special MCp group. This chapter focuses on the nature and structure of finite MC groups.

II. A. Preparatory knowledge

II. A. 1) Basic concepts and conclusions

This section introduces some basic concepts and conclusions.

Definition 1: Let G be a finite noncommutative p-group. If all subgroups of G with index p^t are exchanged, but there exists a subgroup with index p^{t-1} that is non-exchanged, then G is said to be a A_t-group.

Definition 2: Let G be a finite group and M be a commutative subgroup of G. If all subgroups that really contain M are noncommutative, then M is said to be a greatly commutative subgroup of G.

For convenience, we use $ma(G)$ to denote the number of extremely exchanged subgroups of G, $ncma(G)$ to denote the number of acyclic extremely exchanged subgroups of G, and $Ima(G)$ to denote the set consisting of the indices of extremely exchanged subgroups of G.

Definition 3: Let G_1, G_2 be the group, $Z_1 = Z(G_1), Z_2 = Z(G_2)$. Call G_1, G_2 homotopic, denoted $G_1 \sim G_2$, if:

(1) $G_1 / Z_1 \cong G_2 / Z_2$.

(2) $G_1' \cong G_2'$.

(3) One can choose the isomorphism α in (1) and the isomorphism β in (2) such that if $(a_1Z_1)^\alpha = a_2Z_2, (b_1Z_1)^\alpha = b_2Z_2$, then $[a_1, b_1]^\beta = [a_2, b_2]$.

The homotopy relation ~ is an equivalence relation, according to which the group can be divided into a number of equivalence classes, and each class is called a homotopy family.

Definition 4: Let G be a finite non-commutative group, if all the true subgroups of G are commutative, then G is said to be an inner commutative group.

Definition 5: Let G be a finite noncommutative group. If $G'' = 1$ then G is said to be a subcommutative group.

Some of the basic conclusions used in this paper are presented below.

Lemma 1: Let A be a commutative regular subgroup of a noncommutative group G and its quotient group $G / A = \langle xA \rangle$ is cyclic. Then:

(1) The map $a \mapsto [a, x] (a \in A)$ is a full homomorphism of A onto G'.

(2) $G' \cong A / A \cap Z(G)$.

In particular, if the finite noncommutative p-group G has commutative extremal subgroups, then $|G| = p|G'||Z(G)|$.

Lemma 2: Let G be a finite non-commutative p-group, then the following conditions are equivalent:

(1) $|G : Z(G)| = p^2$

(2) $G = HZ(G)$ where H is the inner exchange group.

(3) $Ima(G) = \{p\}$.

Lemma 3: Let G be a finite p-group and A be a commutative extremal subgroup of G. Then:

(1) If $a_1, a_2 \in A, b \in G$, then $[a_1a_2, b] = [a_1, b][a_2, b]$.

(2) Let n be a positive integer. If $a \in A, b \in G$, then:

1) $[a^n, b] = [a, b]^n$.

2) $(ba)^n = b^n a^n \prod_{i=1}^{n-1} [a, ib]^{\binom{n}{i+1}}$.

3) If $d \in G', b \in G - A$, then $(bd)^p = b^p$.

Lemma 4: Let G be a subcommutative group and $a, b \in G, m, n, i, j$ be positive integers. Also let .$[ia, jb] = [a, b, \underbrace{a, \dots, a}_{i-1}, \underbrace{b, \dots, b}_{j-1}]$. Then there are:

$$[a^m, b^n] = \prod_{i=1}^{m} \prod_{j=1}^{n} [ia, jb]^{\binom{m}{i}\binom{n}{j}} \tag{1}$$

Also set $m \geq 2$, then:

$$(ab^{-1})^m = a^m \left(\prod_{i+j \leq m} [ia, jb]^{\binom{m}{i+j}} \right) b^{-m} \tag{2}$$

Lemma 5: Let G be a finite p-group where p is prime. Then $ma(G) \equiv 1 (\bmod p)$.

Lemma 6: Let G be a group of order p^n with cyclic maximal subgroups, $p > 2, n \geq 3$ If G is acyclic, then G has exactly p cyclic maximal subgroups and one acyclic maximal subgroup.

Lemma 7: Let G be a noncommutative finite p group. It has exchange maximal subgroup A. Then the following propositions are equivalent:

(1) $|Z(G)| = p$.

(2) $|G : G'| = p^2$.

(3) G is a group of extremely large class p.

Lemma 8: Let G be a finite p-group. Then the following propositions are equivalent:

(1) G is an inner exchange group.

(2) $d(G) = 2$ and $|G'| = p$.

(3) $d(G) = 2$ and $Z(G) = \Phi(G)$.

Lemma 9: Let G be an inner exchange p-group. Then G is one of the following groups:

(1) Q_8.

(2) $M_p(n, m) = \langle a, b \mid a^{p^n} = b^{p^m} = 1, a^b = a^{1+p^{n-1}} \rangle$, where $n \geq 2, m \geq 1$.

(3) $M_p(n, m, 1) = \langle a, b \mid a^{p^n} = b^{p^m} = c^p = 1, [a, b] = c, [c, a] = [c, b] = 1 \rangle$, where $n \geq m \geq 1$.

In the above representations of groups, groups given by different parameters are not isomorphic to each other, with the exception of a group of type (2) with parameters $p = 2, m = 1, n = 2$ and a group of type (3) isomorphic to the group of type (3) with parameters $p = 2, n = m = 1$, both of which give the dihedral group of order 8 D_8.

Lemma 10: Let G be a maximal class group of order 2^n. Then G is isomorphic to one of the following groups:

(1) Generalized quaternion group: $G = \langle a, b \mid a^{2^{n-1}} = 1, b^2 = a^{2^{n-2}}, [a, b] = a^{-2} \rangle$, where $n \geq 3$.

(2) Dihedral group: $G = \langle a, b \mid a^{2^{n-1}} = b^2 = 1, [a, b] = a^{-2} \rangle$, where $n \geq 3$.

(3) Semi-diahedral group: $G = \langle a, b \mid a^{2^{n-1}} = b^2 = 1, [a, b] = a^{-2+2^{n-2}} \rangle$, where $n \geq 4$.

Lemma 11: Let G be a maximal class 3 group of order 3^n ($n \geq 5$) and G has no exchange maximal subgroups. Then G is one of the following mutually non-isomorphic groups:

(1) n when n is even $2e$:

$$G = \langle s, s_1 \mid s_1^{3^e} = s_2^{3^{e-1}} = 1, s^3 = s_1^{63^{e-1}}, [s_1, s] = s_2, [s_2, s] = s_1^{-3} s_2^{-3}, [s_2, s_1] = s_1^{3^{\grave{o}-1}} \rangle \tag{3}$$

where $\delta = 0, 1, 2$.

(2) When n is an odd number $2e+1$:

$$G = \langle s, s_1 \mid s_1^{3^{\grave{o}}} = s_2^{3^{\grave{o}}} = 1, s^3 = s_2^{\delta 3^{\grave{o}-1}}, [s_1, s] = s_2, [s_2, s] = s_1^{-3} s_2^{-3}, [s_2, s_1] = s_2^{3^{\grave{o}-1}} \rangle \tag{4}$$

where $\delta = 0,1,2$.

Lemma 12: Let G be the group of extremely large classes p of order p^n, where $n \geq 4$ and p is odd prime. It has the exchange maximal subgroup A. Then G is isomorphic to one of the following groups:

(1) Let p be an odd prime and e,r be positive integers, where $1 \leq r \leq p-1$. Let A also be the direct product of r cyclic groups of order p^e and $p-1-r$ cyclic groups of order p^{e-1}, i.e. $A = \langle s_1 \rangle \times \langle s_2 \rangle \times ... \times \langle s_{p-1} \rangle$, where:

$$\circ(s_1) = \circ(s_2) = \ldots = \circ(s_r) = p^e, \circ(s_{r+1}) = \ldots = \circ(s_{p-1}) = p^{e-1} \tag{5}$$

A self-isomorphism of A is defined as follows:

$$\beta : s_1^{\beta} = s_1 s_2, s_2^{\beta} = s_2 s_3, s_{p-2}^{\beta} = s_{p-2} s_{p-1}, s_{p-1}^{\beta} = s_{p-1}^{1-p} s_1^{-p} s_2^{-\binom{p}{2}} \ldots s_{p-2}^{-\binom{p}{p-2}} \tag{6}$$

Let $G = A\tilde{a} \langle \beta \rangle$, then β is a p-order automorphism of A. Let G be a semidirect product of A and $\langle \beta \rangle$, then G is a group of extremely large class p. For $i \geq p$ when $s_{i+1} = s_i^{-1} s_i^{\beta}$ is defined recursively, the sequence $\{\beta, s_1, \ldots, s_i, \ldots\}$ is a chain of G, the consistent elements of G are all of order p and $Z(G) = \langle s_p^{p^{e-1}} \rangle$.

(2) Let p,e,r,A,β be the same as in (1). Let also $G = \langle A,b \rangle$ where $b^p = s_r^{pc-1}$ and b is a self-isomorphism β of A induced on A by conjugate transformations. Then G is the group of extremely large class p and the sequence $\{\beta, s_1, ..., s_i, ...\}$ is a chain of G, and the consistent elements of G are all of order p^2.

(3) Let H be the group in (1), $G = \langle H,a \rangle$, satisfied:

$$a^p = s_1^{-\binom{p}{2}} s_2^{-\binom{p}{3}} ... s_{p-1}^{-1} s_r^{\delta p^{e-1}} (1 \leq \delta \leq p-1) \tag{7}$$

$$\beta^a = \beta s_1^{-1}, s_1^a = s_1 \tag{8}$$

Then G is a group of extremely large class p and the sequence $\{\beta, a, s_1, ..., s_i, ...\}$ is a chain of G, and the consistent elements of G are both of order p and of order p^2. Let $n = re + (p-1-r)(e-1) + 2$, then the class group $|G| = p^n$ has a total of $(n-2, p-1)$ mutually disjoint types.

II. A. 2) Basic conclusions related to non-extremely large exchange subgroups

In this paper, we study finite p-groups with smaller $ma(G)$ and $ncma(G)$. The following important property tells us that such finite p-groups can be studied by their non-commutative subgroups.

Lemma 13: Let G be a finite p-group and H a non-commutative subgroup of G. Then:

(1) $ncma(H) \leq ncma(G)$.

(2) $ma(H) \leq ma(G)$.

Proof: (1) Let $ncma(G) = k$, $A_1, A_2, ..., A_k$ are acyclic extremely commutative subgroups of G. An acyclic extremely commutative subgroup A in $\forall H$ is $\exists A_i$ such that $A \leq A_i$, where $1 \leq i \leq k$. Notice that $A \leq H$. Therefore $A \leq A_i \cap H$ Since A is a great exchange subgroup of H and $A_i \cap H$ is an exchange subgroup, $A_i \cap H \leq A$. Therefore $A = A_i \cap H$. Thus $ncma(H) \leq ncma(G)$.

(2) Let $ma(G) = k$, $A_1, A_2, ..., A_k$ are extremely commutative subgroups of G. A very large exchange subgroup A in $\forall H$ is $\exists A_i$ such that $A \leq A_i$, where $1 \leq i \leq k$. Notice that $A \leq H$. Therefore $A \leq A_i \cap H$. Since A is a great exchange subgroup of H and $A_i \cap H$ is an exchange subgroup, $A_i \cap H \leq A$. Therefore $A = A_i \cap H$. Thus $ma(H) \leq ma(G)$.

The basic properties of a finite noncommutative p-group G of extremely commutative subgroups with commutative extremely commutative subgroups are as follows:

Lemma 14: Let G be a finite noncommutative p-group and A be an exchange extremely subgroup of G. If H is a great exchange subgroup of G and $H \neq A$, then:

(1) $H \cap A = Z(G)$.

(2) The order of H is $p|Z(G)|$.

(3) The elements in $G/Z(G) - A/Z(G)$ are all of order p.

(4) $\max(G) = 1 + \dfrac{|G|}{p|Z(G)|}$.

Notice that the exchange maximal subgroup of a noncommutative group must be its maximal exchange subgroup, but its maximal exchange subgroup need not be its exchange maximal subgroup. An example of a group is first given below.

Example 1: $G=\langle a,b \mid a^{p^2}=b^p=c^p=1,[a,b]=c,[c,a]=1,[c,b]=a^p\rangle$, where $p>2$ is computationally obtained as $|G|=p^4$, $H=\langle b,a^p\rangle$ is an extremely large exchange subgroup of G. Since $|H|=p^2$, H is not an extremely large subgroup of G. Therefore H is not an exchange extremal subgroup of G.

The following simple conclusion gives the relation between exchange extremely subgroups and extremely exchanged subgroups.

Lemma 15: Let G be a finite noncommutative p-group. Then the following conditions are equivalent:

(1) The number of exchange extremely subgroups of G is equal to the number of extremely exchange subgroups of G.

(2) The number of exchange maximal subgroups of G is greater than one.

(3) The number of exchange maximal subgroups of G is $1+p$.

Proof: (1)⇒(2): since G is a finite noncommutative p-group, all $ma(G)\geq 1+p$ by Lemma 5. Since the number of exchange maximal subgroups of G is equal to the number of maximal exchange subgroups of G, the number of exchange maximal subgroups of G is greater than 1.

(2)⇒(3): by the condition let A,B be two distinct exchange maximal subgroups of G. Then $|A|=|B|$. Let H be any maximal exchange subgroup of G. If $H\neq A$, $|H|=|B|=p|Z(G)|$ by Lemma 14(2). If $H\neq B$, the same holds for $|H|=|A|=p|Z(G)|$. Thus $|A|=|B|=|H|=p|Z(G)|$. Therefore the order of the great exchange subgroup of G is equal to the order of its exchange great subgroup. Thus the great exchange subgroup of G is the exchange maximal subgroup of G. Therefore the number of exchange maximal subgroups of G is equal to the number of exchange maximal subgroups of G. Thus by Lemma 14(4) $ma(G)=1+\frac{|G|}{p|Z(G)|}=1+p$. That is, the number of exchange maximal subgroups of G is $1+p$.

(3)⇒(1): the conclusion follows from the proof procedure (2)⇒(3).

Corollary 1: Let G be a finite non-commutative p-group. Then the number of exchange maximal subgroups of G is 0, 1 or $1+p$.

Proof: if G has no exchanged maximal subgroups, then the number of exchanged maximal subgroups of G is 0. If G has only one exchanged maximal subgroup then the number of exchanged maximal subgroups of G is 1 - if the number of exchanged maximal subgroups of G is greater than 1, then by Lemma 15 the number of exchanged maximal subgroups of G is $1+p$.

II. B.Finite MC groups

This section gives some properties and carve-outs of MC_p groups.

First, we give some equivalence conditions for MC groups.

Theorem 1: Let G be a finite noncommutative group. Then the following propositions are equivalent:

(1) G is an MC group.

(2) For any binary generated noncommutative subgroup H in G, there is $C_G(H)=Z(G)$.

(3) For any inner exchange subgroup H in G, there is $C_G(H)=Z(G)$.

(4) For any non-commutative subgroup H in G, there is $Z(H)\leq Z(G)$.

(5) For any $x\in G\setminus Z(G)$, there is $C_G(x)$ which is an exchange group.

(6) For any subgroup H in G and satisfying $H \nleq Z(G)$, all $C_G(H)$ are exchange groups.

(7) For any noncommutative subgroups A and B in G and satisfying $A\leq B$, both have $Z(A)\leq Z(B)$.

(8) For any two distinct extremely commutative subgroups A and B in G, there is $A\cap B=Z(G)$.

Corollary 2 follows from Theorem 1(7): let G be the MC group. Then every non-commutative subgroup in G is also a MC group.

Theorem 2: Let G be a finite noncommutative group. Then G is MC group if and only if the centralized subgroup of G is a modular lattice of length 2.

The MC group can also be inscribed in a graph-theoretic way. For each finite noncommutative group G, let $\Gamma(G)=(V,E)$ be a simple undirected graph where: $V=\{Z(G)x \mid x\in G\setminus Z(G)\}$ and $E=\{\{Z(G)x,Z(G)y\} \mid [x,y]=1, xy^{-1}\notin Z(G)\}$.

By Theorem 1(5), G is an MC group if and only if for any $x\in G\setminus Z(G)$, there is $C_G(x)$ which is the exchange group. Clearly, for any $x\in G\setminus Z(G)$, there is $C_G(x)$ which is the exchange group if and only if the derived subgraph of the neighborhood of each vertex of $\Gamma(G)$ is a complete graph. Thus the following conclusion can be obtained:

Proposition 1: Let G be a finite noncommutative group, then G is a MC group if and only if every connected branch of $\Gamma(G)$ is a complete map.

Lemma 16: Let G and H be two finite noncommutative groups. If G and H are homotopic, then $\Gamma(G)\cong\Gamma(H)$.

This follows from Proposition 1 and Lemma 16:

Theorem 3: Let G and H be two finite noncommutative groups. If G and H are homotopic, then G is MC group if and only if H is MC group.

Corollary 3: Let G be a finite noncommutative group and H a finite group. Then $G*H$ is an MC group if and only if G is an MC group and H is an exchange group.

Proof:(⇐) G is homotopic to $G*H$. Since G is an MC group, it follows from Theorem 3 that $G*H$ is also an MC group.

(⇒) By Corollary 2 it follows that G is MC group. Let $K=G*H$. Since G is a non-commutative group, there exists an element $y\in G/Z(K)$. Clearly, $H\leq C_K(y)$. By Theorem 1(5), $C_K(y)$ is the exchange group. Therefore H is the exchange group.

The MCp-group is studied next. The following proposition states that the MC_p-group is a fairly large class of groups.

(1) The group with exchange extremely large subgroups.

(2) The group whose centers of all noncommutative subgroups are equal.

(3) The group satisfying $|G:Z(G)|\leq p^3$.

(4) The group of all exchange subgroups H satisfying $HZ(G)/Z(G)$ is a cyclic group, then G is an MC group.

Proof:(1)~(2) follow directly from the previous theorem.

Since G is a non-commutative group, $|G:Z(G)|=p^2$ or p^3. If $|G:Z(G)|=p^2$, then G has exchange extremely large subgroups. It also follows from (1) that G is an MC group. Suppose $|G:Z(G)|=p^3$. Let H be a noncommutative subgroup of G and $L=HZ(G)$. Then and:

$$Z(G)\leq Z(L) \text{ and } Z(H)\leq Z(L) \tag{9}$$

If $L=G$, then $Z(H)\leq Z(G)$. If $L<G$, then $p^2\leq|L:Z(L)|\leq|L:Z(G)|\leq p^2$. Thus $Z(L)=Z(G)$ and $Z(H)\leq Z(G)$. So by Theorem 1(4), G is an MC group.

$\bar{G}=G/Z(G)$ is a dihedral group or a group with square number p. Suppose that A and B are two different extremely commutative subgroups in G. From the problem and the correspondence theorem, $\bar{A}=A/Z(G)$ and $\bar{B}=B/Z(G)$ are two distinct extremely cyclic subgroups in $\bar{G}$. Thus $\bar{A}\cap\bar{B}=\overline{A\cap B}=\bar{1}$. Thus $A\cap B\leq Z(G)$ can be obtained. Clearly, $Z(G)\leq A\cap B$. So $A\cap B=Z(G)$. Also by Theorem 1(8), G is an MC group.

Lemma 17: Let G be the group MC_p. Then:

(1) If A is a great exchange subgroup of G, then $|A|\geq|G:G'|$.

(2) $Phi(G)Z_{c-1}(G)$ is the exchange group where $c=cl(G)$. In particular, G is a subcommutative group.

Proof:(1) Let $x\in A/Z(G)$. Then $A\leq C_G(x)$. Since G is an MC group, it follows from Theorem 1(5) that $C_G(x)$ is an exchange group. Thus by the greatness of A it follows that $A=C_G(x)$. Obviously:

$$|G:C_G(x)|=|x^G|\leq|xG'|=|G'| \tag{10}$$

So $|A|=|C_G(x)|\geq|G:G'|$.

(2) First prove that $\Phi(G)$ is the exchange group. Clearly, $K_c(G) \le Z(G)$ and $K_c - 1(G)$ ' $Z(G)$. Let $x \in K_{c-1}(G)/Z(G)$ and $x^p \in Z(G)$. Then $[x^p, g] = 1$ for any $g \in G$. Since $[x,g] \in K_c(G)$ and $K_c(G) \le Z(G)$, it follows:

$$[x, g^p] = [x, g]^p = [x^p, g] = 1 \tag{11}$$

Thus $[x,\square_1(G)] = 1$ can be obtained. On the other hand:

$$[K_{c-1}(G), K_2(G)] \le K_{c+1}(G) = 1 \tag{12}$$

This implies that $[x, K_2(G)] = 1$, and because $\Phi(G) = G' U_1(G)$, $[x, \Phi(G)] = 1$. Notice that $x \in K_{c-1}(G) \le \Phi(G)$. Then $x \in Z(\Phi(G))$. Thus $Z(\Phi(G))$ œ $Z(G)$. So by Theorem 1(4) $\Phi(G)$ is the exchange group. Thus G' is the exchange group, i.e., G is subexchange group.

Then prove that $\Phi(G)Z_c - 1(G)$ is the exchange group. Obviously:

$$[K_{c-1}(G), Z_{c-1}(G)] = 1 \text{ and } [K_{c-1}(G), K_2(G)] = 1 \tag{13}$$

If G has exchange extremely large subgroups A, then:

$$[K_{c-1}(G), U_1(G)] \le [A, A] = 1 \tag{14}$$

If G has no exchange of extremely large subgroups, then $U_1(G) \le Z(G)$ by Theorem 1(7). Thus $[K_{c-1}(G),\square_1(G)] = 1$. Thus we can obtain $[K_c - 1(G), \Phi(G)] = 1$. So:

$$K_{c-1}(G) \le C_G(\Phi(G)Z_{c-1}(G)) \tag{15}$$

Notice that $K_c - 1(G)$ ' $Z(G)$. Then it follows from G being a MC group that $\Phi(G)Z_c - 1(G)$ is an exchange group.

Call the MCp group G the MC_1 group, and next study the properties of the MC_1 group.

Lemma 18: Let G be the group MC_1 and $Z(G) \le G'$. Then:

(1) $|G| \le |G'|^3$.

(2) $U_1(G) \le Z(G) \le G' = \Phi(G)$. If $p = 2$ or $cl(G) = 2$, then G is a special p-group.

Proof: (1) Let A and B be two distinct extremely commutative subgroups in G. Then by Theorem 1(8) we have $A \cap B = Z(G)$. Also by Lemma 17(1):

$$|G| \ge |AB| = \frac{|A||B|}{|A \cap B|} \ge \frac{|G:G'|^2}{|Z(G)|} \ge \frac{|G:G'|^2}{|G'|} = \frac{|G|^2}{|G'|^3} \tag{16}$$

So $|G| \le |G'|^3$.

(2) By Theorem 1(7), we have $\square_1(G) \le Z(G)$. Since $Z(G) \le G'$, $G' = \Phi(G)$. If $p = 2$, then $\square_1(G) = \Phi(G)$. Thus it follows that $\square_1(G) = Z(G) = G' = \Phi(G)$. So G is a special 2-group. If $cl(G) = 2$, then $G' \le Z(G)$. Further $Z(G) = G' = \Phi(G)$. So G is a special p-group.

The carve-out of the MCp group is given:

Theorem 4: Let G be a finite noncommutative p-group. Then G is a MC group if and only if G is homotopic to a p-group with exchange maximal subgroups or homotopic to a subexchangeable MC_p group H satisfying $\square_1(H) \le Z(H) \le H' = \Phi(H)$. In particular, if $p = 2$ or $cl(G) = 2$, then G is an MC group if and only if G is homotopic to the p-group with exchange maximal subgroups or to the special MC_p group.

Proof:(⇒) Let H be the base group in the family of homotopies containing the group G, then it follows that $Z(H) \le H'$. Since G is an MC group, it follows from Theorem 3 that H is also an MC group. If H has no exchange of extremely large subgroups, then H is the group MC_1. It also follows from Lemma 17(2) and Lemma 18(2) that H is a subexchange group satisfying $V_1(H) \le Z(H) \le H' = \Phi(H)$. In particular, H is a special group if H is a 2-group or $cl(H) = 2$.

(⇐) The conclusion holds by Proposition 2(1).

II. C.Specialized MC groups

In this section, we will give a criterion for determining that a special p-group is a MC group and provide a method for homotopy classifying special MC_p-groups. Clearly, for any special p-group G, there is $cl(G) = 2$ and

$\exp(G/G')=p$. Let G be a finite p-group with $cl(G)=2$ and $\exp(G/G')=p$. Then G' and G/G' are primitive exchange groups. Therefore, it can be set up:

$$\bar{G}=G/G'=\langle\bar{a}_1\rangle\times\langle\bar{a}_2\rangle\times\cdots\times\langle\bar{a}_n\rangle \text{ and } G'=\langle t_1\rangle\times\langle t_2\rangle\times\cdots\times\langle t_r\rangle \tag{17}$$

Then $G=\langle a_1,a_2,\cdots,a_n\rangle$. Since $cl(G)=2$, it follows:

$$[a_i,t_u]=[t_l,t_u]=1, 1\le i\le n, 1\le u,l\le r \tag{18}$$

Suppose $[a_i,a_j]=\prod_{n=1}^{r}t_u^{x_{uij}}, x_{uij}\in F_p, 1\le i,j\le n$. Then $X_u=(x_{uij})_{n\times n}$ is the antisymmetric matrix over the domain F_p, where $1\le u\le r$.

In this case, $X_1,X_2,\cdots,X_r$ are all nonzero matrices. Call $n\times nr$ matrices:

$$X=(X_1,X_2,\cdots,X_r) \tag{19}$$

Is the permutation submatrix of G.

For any $A\in F_p^{l\times n}, B\in F_p^{n\times m}$ and $C=(c_{ij})_{r\times r}\in F_p^{r\times r}$, notation:

$$A\Diamond X\Diamond B=A\Diamond(X_1,X_2,\cdots,X_r)\Diamond B=(AX_1B,AX_2B,\cdots,AX_rB) \tag{20}$$

And:

$$X\circ C=(X_1,X_2,\cdots,X_r)\circ C=\left(\sum_{k=1}^{r}c_{k1}X_k,\sum_{k=1}^{r}c_{k2}X_k,\cdots,\sum_{k=1}^{r}c_{kr}X_k\right) \tag{21}$$

The operation $\Diamond$ can be viewed as a multiplication of partitioned matrices, and the operation $\circ$ can be viewed as a multiplication of formal partitioned matrices. It is easy to verify that $E\Diamond X\Diamond B=X\Diamond B, A\Diamond X\Diamond E=A\Diamond X, E\Diamond X\Diamond E=X, X\circ E=X$ and $X\circ C\circ C^{-1}=X$.

First, the carve-out of the special p-group is given.

Proposition 3: Let G be a finite p-group of power zero class 2, $\exp(G/G')=p$ and $X\in F_p^{n\times nr}$ is a commutative submatrix of G. Then G is a special p-group if and only if $rank(X)=n$.

Proof: by the definition of the special p-group, it is sufficient to prove:

$$Z(G)\le G'\Leftrightarrow \operatorname{rank}(X)=n \tag{22}$$

Let $g,h\in G$ such that $\bar{g}=\prod_i=1^n\bar{a}_i^{k_i}$ and $\bar{h}=\prod_j=1^n\bar{a}_j^{l_j}$, where:

$$k=(k_1,k_2,\cdots,k_n)^T, l=(l_1,l_2,\cdots,l_n)^T\in F_p^n \tag{23}$$

Notice that $G'\le Z(G)$ and $\exp(G/G')=p$. Then it can be obtained:

$$\begin{aligned}[g,h]&=\prod_{i=1}^{n}a_i^{k_i},\prod_{j=1}^{n}a_j^{l_j}=\prod_{i,j=1}^{n}[a_i,a_j]^{k_il_j}\\&=\prod_{i,j=1}^{n}\prod_{u=1}^{r}t_u^{x_{uij}k_il_j}=\prod_{u=1}^{r}\prod_{i,j=1}^{n}t_u^{x_{uij}k_il_j}=\prod_{u=1}^{r}t_u^{\sum_{j=1}^{n}x_{uij}k_il_j}\end{aligned} \tag{24}$$

$$[g,h]=1\Leftrightarrow\sum_{i,j=1}^{n}x_{uij}k_il_j=0, 1\le u\le r\Leftrightarrow k^T\Diamond X\Diamond l=0 \tag{25}$$

And so it can be obtained:

$$\begin{aligned}g\in Z(G)\quad&\Leftrightarrow\quad\text{For all } h\in G\text{, there is }[g,h]=1\\&\Leftrightarrow\quad\text{For all } l\in F_p^n\text{, there is } k^T\circ X\Diamond l=0\\&\Leftrightarrow\quad k^TX=0\end{aligned} \tag{26}$$

So $Z(G)\le G'\Leftrightarrow$ the system of chi-square linear equations $X^Tx=0\Leftrightarrow$ has only zero solutions in F_p^n $\Leftrightarrow rank\left(X\right)=n$, so the conclusion holds.

Then, the carve-out of the special MCp-group is given.

Theorem 5: Let G be a finite special p-group and $X\in F_p^{n\times nr}$ be a permutation submatrix of G. Then G is an MC group if and only if for any $k,l\in F_p^n$ and satisfying $k^T\circ X\circ l\ne 0$ there is:

$$\operatorname{rank}(X\Diamond k, X\Diamond l)=n \tag{27}$$

Proof: by Theorem 1(2), G is an MC group if and only if for any $g,h\in G$ and $[g,h]\neq 1$, there is $C_G(\langle g,h\rangle)=Z(G)$.

Let $g,h\in G$ such that $\overline{g}=\prod_{i=1}^{n}\overline{a}_i^{k_i}$ and $\overline{h}=\prod_{j=1}^{n}\overline{a}_j^{l_j}$, where:

$$k=(k_1,k_2,\cdots,k_n)^T, l=(l_1,l_2,\cdots,l_n)^T\in F_p^n \tag{28}$$

Then there is a proof of the process (*) by Proposition 3:

$$[g,h]\neq 1\Leftrightarrow k^T\Diamond X\Diamond l\neq 0 \tag{29}$$

Suppose $[g,h]\neq 1$. Let $b\in G$ such that $\overline{b}=\prod_{i=1}^{n}\overline{a}_i^{b_i}$ and $b=(b_1,b_2,\cdots,b_n)^T\in F_p^n$. Then we have by (*):

$$\begin{aligned} b\in C_G(\langle g,h\rangle) &\Leftrightarrow [b,g]=[b,h]=1\\ &\Leftrightarrow b^T\Diamond X\Diamond k=0 \text{ and } b^T\Diamond X\Diamond l=0\\ &\Leftrightarrow b^T(X\Diamond k,X\Diamond l)=0\end{aligned} \tag{30}$$

Note that $G^{'}=Z(G)$. Is then available:

$$\begin{aligned}&C_G(\langle g,h\rangle)=Z(G)\Leftrightarrow \text{Homogeneous linear equation system}\\ &(X\Diamond k,X\Diamond l)^T x=0 \quad \text{Only the zero solution in } F_p^n\\ &\Leftrightarrow rank(X\Diamond k,X\Diamond l)=n\end{aligned} \tag{31}$$

So:

$$G \text{ is the MC group} \Leftrightarrow \text{For any } k,l\in F_p^n \text{ and } k^T\circ X\circ l\neq 0, \text{ there is } (X\Diamond k,X\Diamond l)=n \tag{32}$$

Therefore the conclusion holds.

A sufficiently necessary condition for two special p-groups to be homotopic is as follows:

Proposition 4: Let G and H be two finite special p-groups, and $X,Y\in F_p^{n\times nr}$ be the permutation submatrices of G and H respectively. Then G and H are homotopic if and only if there exist two nonsingular matrices $C\in F_p^{n\times n}$ and $D\in F_p^{r\times r}$ such that:

$$(C^T\Diamond Y\Diamond C)\circ D=X \tag{33}$$

Proof: let G and H be special p-groups such that:

$$\overline{H}=H/H^{'}=\langle\overline{b}_1\rangle\times\langle\overline{b}_2\rangle\times\cdots\times\langle\overline{b}_n\rangle \text{ and } H^{'}=\langle s_1\rangle\times\langle s_2\rangle\times\cdots\times\langle s_r\rangle \tag{34}$$

Then $H=\langle b_1,b_2,\cdots,b_n\rangle$. Assumptions:

$$[b_i,b_j]=\prod_{u=1}^{r}s_u^{y_{uij}}, y_{uij}\in F_p, 1\leq i,j\leq n, Y_u=(y_{uij})_{n\times n} \text{ and } Y=(Y_1,Y_2 \tag{35}$$

Therefore $H^{'}=Z(H)$ and $G^{'}=Z(G)$.

If G is homotopic to H, the following isomorphic mapping exists:

$$\begin{aligned}\alpha:\ G/Z(G) &\rightarrow H/Z(H)\\ Z(G)a &\mapsto Z(H)a^{'}\end{aligned} \quad \text{and} \quad \beta: G^{'}\rightarrow H^{'} \tag{36}$$

such that $[a,b]^{\beta}=[a^{'},b^{'}]$ where $a,b\in G$. Assumptions:

$$(Z(G)a_i)^{\alpha}=Z(H)\prod_{j=1}^{n}b_j^{a_{ji}} \text{ and } t_i^{\beta}=\prod_{j=1}^{r}s_j^{b_{ji}} \tag{37}$$

Let $A=(a_ij)_{n\times n}\in F_p^{n\times n}$ and $B=(b_ij)_{r\times r}\in F_p^{r\times r}$. Then it follows that A and B are nonsingular matrices. Obviously:

$$[a_i,a_j]^{\beta}=\left(\prod_{u=1}^{r}t_u^{x_{uij}}\right)^{\beta}=\prod_{u=1}^{r}(t_u^{\beta})^{x_{uij}}=\prod_{u=1}^{r}\left(\prod_{l=1}^{r}s_l^{b_{lu}}\right)^{x_{uij}}=\prod_{u,l=1}^{r}s_l^{b_{lu}x_{uij}}=\prod_{l=1}^{r}s_l^{\sum_{u=1}^{r}b_{lu}x_{uij}} \tag{38}$$

Note that $Z(H)a_i^{'}=Z(H)\prod_{k=1}^{n}b_k^{a_{ki}}$. Is then available:

$$\left[a_i^{'},a_j^{'}\right]=\left[\prod_{k=1}^{n}b_k^{a_{ki}},\prod_{m=1}^{n}b_m^{a_{mj}}\right]=\prod_{k,m=1}^{n}[b_k,b_m]^{a_{ki}a_{mj}}=\prod_{k,m=1}^{n}\left(\prod_{l=1}^{r}s_l^{y_{lkm}}\right)^{a_{ki}a_{mj}}$$
$$=\prod_{k,m=1}^{n}\prod_{l=1}^{r}s_l^{y_{lkm}a_{ki}a_{mj}}=\prod_{l=1}^{r}\prod_{k,m=1}^{n}s_l^{y_{lkm}a_{ki}a_{mj}}=\prod_{l=1}^{r}s_l^{\sum_{k,m=1}^{n}y_{lkm}a_{ki}a_{mj}} \tag{39}$$

So:

$$[a_i,a_j]^{\beta}=[a_i^{'},a_j^{'}] \Leftrightarrow \prod_{l=1}^{r}s_l^{\sum_{u=1}^{r}b_{lu}x_{uij}}=\prod_{l=1}^{r}s_l^{\sum_{k,m=1}^{n}y_{lkm}a_{ki}a_{mj}}$$
$$\Leftrightarrow \text{For any } 1\le l\le r, \text{ there is } \sum_{u=1}^{r}b_{lu}x_{uij}=\sum_{k,m=1}^{n}y_{lkm}a_{ki}a_{mj} \tag{40}$$
$$\Leftrightarrow X\circ B^T=A^T\circ Y\circ A$$
$$\Leftrightarrow X=(A^T\Diamond Y\Diamond A)\circ B^{-T}$$

Let $C=A$ and $D=(B^{-1})^T$. Then C and D are nonsingular matrices and:

$$X=(C^T\circ Y\circ C)\circ D \tag{41}$$

Suppose $C=(a_{ij})_{n\times n}\in F_p^{n\times n}$ and $(D^{-1})^T=(b_{ij})_{r\times r}\in F_p^{r\times r}$ defines two mappings:

$$\alpha: G/Z(G)\to H/Z(H) \text{ and } \beta: G^{'}\to H^{'} \tag{42}$$

Make:

$$(Z(G)a_i)^{\alpha}=Z(H)\prod_{j=1}^{n}b_j^{a_{ji}} \text{ and } t_i^{\beta}=\prod_{j=1}^{r}s_j^{b_{ji}} \tag{43}$$

It is easy to verify that α and β are isomorphic mappings. Further, for any $a,b\in G$, suppose:

$$\bar{a}=\prod_{i=1}^{n}\bar{a}_i^{k_i} \text{ and } \bar{b}=\prod_{j=1}^{n}\bar{a}_j^{l_j} \tag{44}$$

Then:

$$[a,b]^{\beta}=\left(\prod_{i=1}^{n}a_i^{k_i},\prod_{j=1}^{n}a_j^{l_j}\right)^{\beta}=\prod_{i,j=1}^{n}([a_i,a_j]^{k_il_j})^{\beta}=\prod_{i,j=1}^{n}([a_i,a_j]^{\beta})^{k_il_j} \tag{45}$$

Let $Z(H)a^{'}=(Z(G)a)^{\alpha}$ and $Z(H)a_i^{'}=(Z(G)a_i)^{\alpha}$. Then $Z(H)a^{'}=Z(H)\prod_{i=1}^{n}(a_i^{'})^{k_i}$.

Similarly, there is $Z(H)b^{'}=Z(H)\prod_{j=1}^{n}(a_j^{'})^{l_j}$. Thus:

$$\left[a^{'},b^{'}\right]=\left[\prod_{i=1}^{n}(a_i^{'})^{k_i},\prod_{j=1}^{n}(a_j^{'})^{l_j}\right]=\prod_{i,j=1}^{n}\left[a_i^{'},a_j^{'}\right]^{k_il_j} \tag{46}$$

From the necessity of the proof: $X=(C^T\Diamond Y\Diamond C)\circ D\Leftrightarrow [a_i,a_j]^{\beta}=[a_i^{'},a_j^{'}]$. where $1\le i,j\le n$. Thus $[a,b]^{\beta}=[a^{'},b^{'}]$. This leads to G and H being homotropic.

III. Calculation of critical exponents for noncommutative maximal subgroups of finite groups

III. A. Calculation of critical exponents for noncommutative extremely large subgroups

III. A. 1) Two lower bounds on the number of noncommutative maximal subgroups

In this section, we study the effect of informal Sylow subgroups on the number of noncommutative maximal subgroups of a finite nonidempotent zero group, mainly in terms of solvability and non-solvability, and the lemmas, theorems and examples obtained are as follows:

Lemma 19: Let G be a finite group and $p\in\pi(G)$. If N is a solvable minimal regular subgroup of G, then N is a primitive exchange p-group.

Lemma 20: Let G be a non-solvable finite group, and m and n be the maximal prime and minimal soliton in $\pi(G)$, respectively, then we have $m\ge|\pi(G)|+n$.

Proof: it is clear that any two consecutive prime numbers differ by at least two digits except between 2 and 3. Therefore $m-n \geq 2(|\pi(G)|-2)+1$. Also by Burnside's $p^a q^b -$ theorem, we know that $|\pi(G)| \geq 3$, if not G is solvable contradicting the assumption. Thus $m-n \geq |\pi(G)|$ is $m \geq |\pi(G)|+n$.

Lemma 21: Let N be a $p-$ solvable regular subgroup of the finite group G. If H is a Hall $p^{'}-$ subgroup of N, then $G = N_G(H)N$.

Proof: since N is $p-$ solvable, H must exist. Let $g \in G$, it is clear that $H^g \subseteq N^g = N$. Thus H^g is a Hall $p^{'}-$ subgroup of N. Also N is $p-$ solvable, so H^g and H are $N-$ conjugate i.e. there exists $n \in N$ satisfying $H^{gn} = H$. In this case, there is $gn \in N_G(H)$ i.e. $g \in N_G(H)n^{-1} \subseteq N_G(H)N$. Therefore, there is $G \subseteq N_G(H)N$ and hence $G = N_G(H)N$.

Theorem 6: Let N be a noncommutative minimal regular subgroup of the finite group G and p be the largest prime number in $\pi(N)$. Let $N = R_1 \times R_2 \times \cdots \times R_l$ be the direct product of isomorphic mono-groups, then there are p^l at least p^l non-idempotent zero extremal subgroups of G which do not contain N.

Proof: let P be a Sylow $p-$ -subgroup of N. By Frattini assertion, there is $G = NN_G(P)$. Also $Z(J(P))charJ(P)$, $J(P)charP$, and so $Z(J(P))charP$. Also $P (N_G(P)$, so $Z(J(P)) (N_G(P)$, so $N_G(P) \subseteq N_G(Z(J(P)))$. Clearly, $N_G(Z(J(P))) \neq G$. If not, then we have $Z(J(P)) (G$ is a regular subgroup of G contained in N, contradicting the minimal regularity of N.

Let M be an extremely small subgroup of G containing $N_G(Z(J(P)))$. Then:

$$G = NN_G(P) = NN_G(Z(J(P))) = NM \tag{47}$$

If M is idempotent zero, then $N_G(Z(J(P)))$ is idempotent zero and hence $N_N(Z(J(P)))$ is idempotent zero. If $p=2$, then N is a 2-group, and by the relevant lemma N is an exchange group, contradicting the assumption. If $p>2$, N is $p-$ power zero by the Glauberman-Tompson regular $p-$ complement theorem, contradicting the minimal regularity of N. Clearly $N Ú M$, if or not $G = NM \subseteq M$ contradicts it. Thus M is a nonidempotent zero extremal subgroup of G that does not contain N.

Let $S_i = R_i \cap M, i = 1,2,\cdots,l$. Since M passes actions on $\{R_1, R_2, \cdots, R_l\}$, it follows that S_i and S_j are $M-$ conjugate with $i,j = 1,2,\cdots,l$. Also N is a direct product of isomorphic unigroups, so $R_i Ú M, i = 1,2,\cdots,l$ while $R_i \neq S_i \neq 1$ and S_i is non-regular in R_i. Thus M is a self-regularizing extremely large subgroup of G such that the R_i left-multiplication acts on the set consisting of the left-chaperone sets of all S_i in R_i. Thus R_i is isomorphic to a subgroup of the symplectic group $A_{|R_i:S_i|}$. Thus $p \leq |R_i : S_i|$. Let T_i be the image of the projection map from $N \cap M$ to R_i. Since $R_i (N$, it follows that $S_i (N \cap M$ and therefore $S_i (T_i$. Again R_i is a unitary group, so T_i is a true subgroup of R_i. Thus $U = T_1 \times T_i \times \cdots \times T_i$ is a true subgroup of N. Thus M regularizes U and UM is a true subgroup of G. Again M is extremal, so $U \subseteq M$ and $T_i \subseteq S_i$. Clearly $S_i \subseteq T_i$, hence $S_i = T_i$. And so:

$$|G:M| = |N : N \cap M| = |R_1 \times R_2 \times \cdots \times R_l : S_1 \times S_2 \times \cdots \times S_l| \geq p^l \tag{48}$$

In summary, there are at least p^l non-idempotent zero maximal subgroups in G that do not contain N.

Theorem 7: Let G be a finite non-solvable group, then there is a Sylow $q-$ subgroup of $|n(G)| \geq |\pi(G)|+p$ where $p = \min\{q \in \pi(G) | G$ that is not regular in G.

Proof: take a minimal regular subgroup N of G when N is non-solvable, such that q is the largest prime in $\pi(N)$, and by Theorem 6, there are at least q non-idempotent zero maximal subgroups in G that do not contain N. By Tompson's odd order theorem, $|N$ and $|G|$ are even. Let M be a Sylow 2-subgroup of G. If M is a regular subgroup of G, then $M \cap N \lhd G$ and $M \cap N = N$ is a power zero group contradicting the assumption. Thus $p = 2-$ simultaneously, it follows that $|m(G/N)| \geq \pi(G/N)$. Also N is a non-solvable group, so all extremely large subgroups of G containing N are non-idempotent zero. Thus we have $|n(G)| \geq |\pi(G/N)|+q$. Clearly, $|\pi(G/N)| \geq |\pi(G)|-|\pi(N)|$, which gives $q \geq |\pi(N)|+2$. To summarize, $|n(G)| \geq |\pi(G)|+2$.

when N is solvable. It is known that N is a primitive exchange $r-$ group, where r is some prime number in $\pi(G)$. Also, G/N is a nonsolvable group. If $\pi(G/N)=\pi(G)$, by induction, we have $|n(G/N)|\geq|\pi(G/N)|+p$, and so $|n(G)|\geq|\pi(G)|+p$. If $\pi(G/N)=\pi(G)-\{r\}$, N is the unique Sylow $r-$ subgroup of G. By induction, $|n(G/N)|\geq|\pi(G)|-1+p$. By the Schur-Zassenhaus theorem, N has a complement in G, denoted H, and $H\cong G/N$. Thus by the non-solvability of G/N, we get the non-idempotent zero of H, and it is clear that H is an extremely large subgroup of G that does not contain N, so there are non-idempotent zero extremely large subgroups in G that do not contain N. Thus $|n(G)|\geq|\pi(G)|+p$, where $p=\min\{q\in\pi(G)|G$ of the Sylow $q-$ subgroups in G is nonregular.

Theorem 8: It does not necessarily hold for solvable groups. For example, power zero groups, inner power zero groups. And we have:

Example 2: Let $p>1$ be prime, $m|p-1$ and $|\pi(m)|\geq 2$, then there exists a finite solvable group G of order pm that is not power zero and not inner power zero such that $|n(G)|\leq|\pi(G)|-1$.

Proof: from the related literature, if $p>1$ is prime and $m|p-1$ and $m>1$, then there exists a finite group G of order pm and $Z(G)=1$, where the Sylow $p-$ subgroup P of G is regular in G and G/P is a cyclic group Obviously, by $Z(G)=1$ and $|\pi(m)|\geq 2$, G is a non-idempotent zero, a non-internal idempotent zero. Also G/P is a cyclic group, so the number of extremely large subgroups of G/P is $|\pi(G/P)|$, and the number of extremely large subgroups of G containing P is $|\pi(G/P)|$.

Taking any extremely large subgroup M that does not contain P in G, then $G=PM$ and $P\cap M=1$. Thus, $M\cong G/P$ is a cyclic group and hence M is idempotent zero. In summary, $|n(G)|\leq|\pi(G)|-1$.

III. A. 2) Lower bound on the number of non-commutative maximal subgroups

In this section, we study the effect of informal Sylow subgroups on the number of noncommutative maximal subgroups of a finite nonidempotent zero group, and obtain the following lemma and theorem:

Lemma 22: Let G be a finite group and q be the largest prime in $\pi(G)$. If $Q\in Syl_q(G)$ is not regular in G, then there is at least $q+1$ extremely large subgroups of G containing $N_G(Q)$

Proof: $N_G(Q)\neq G$ by the fact that Q is not regular in G. It is sufficient to set M to be a very large subgroup of G containing $N_G(Q)$ and $n=|G:M|$. Clearly, M is a self-regularizing subgroup of G. Let the G right multiplication act on the set consisting of all right companion sets of M in G, then $G/Core_G(M)$ is isomorphic to a subgroup of the symmetric group S_n with number n. If $n<q$, then $Q\leq Core_G(M)$ is known by the fact that q is the largest prime in $\pi(G)$. Thus, by the Frattini assertion $G=Core_G(M)N_G(Q)=M$, a contradiction. On the other hand by $Q\subseteq M$ it follows that $n\neq q$, and so $n>q$. Thus there are at least $q+1$ extremely large subgroups of G containing $N_G(Q)$.

Theorem 9: If G is a finite nonidempotent zero group, then there is a $|m(G)|\geq|\pi(G)|+p$ where $p=\min\{q\in\pi(G)|G$ of Sylow $q-$ subgroups of G that are not regular in G}.

Proof: take the minimal regular subgroup N of G. Assume that N is non-solvable, and if G/N is a non-idempotent zero group, then by the induction hypothesis there are at least $|\pi(G/N)|$ very large subgroups of G/N. If G/N is a power zero group, then G/N also has at least $|\pi(G/N)|$ extremely large subgroups. Also $|\pi(G/N)|\geq|\pi(G)|-|\pi(N)|$, so there are at least $|\pi(G)|-|\pi(N)|$ extremely large subgroups in G/N. Thus there are at least $|\pi(G)|-|\pi(N)|$ extremely large subgroups of G containing N.

Since N is a noncommutative minimal regular subgroup of G, N is a direct product of isomorphic noncommutative unigroups. It also follows from FeitTompson's Lemma that N must be of even order, and hence G is also of even order. Moreover, the Sylow 2-subgroup of G must be informal in G. Thus $p=2$. Let q be the largest prime in $\pi(N)$, and by Theorem 6 there are at least q extremely large subgroups in G which do not contain N.

By Lemma 20, $q\geq|\pi(N)|+2$, so there is:

$$|m(G)|\geq|\pi(G)|-|\pi(N)|+q\geq|\pi(G)|+2 \tag{49}$$

Theorem proved.

Now assume that N is a solvable group. Thus N is a primitive exchange $t-$ group, where t is some prime number in $\pi(G)$. Assuming that G/N is non-idempotent zero, it follows from the induction hypothesis that there are at least $|\pi(G/N)|+p_1$ extremely large subgroups of G/N, where $p_1=\min\{q\in R\pi(G/N)|G/N$ of Sylow $q-$ subgroups in G/N that are not regular}. Clearly, $p_1\geq p$ and $|\pi(G/N)|\geq|\pi(G)|-1$.

Thus, G/N has at least $|\pi(G/N)|+p$ extremely large subgroups. Thus G has at least $|\pi(G/N)|+p$ extremely large subgroups containing N.

If $|\pi(G/N)|=|\pi(G)|$, the theorem is proved. If $|\pi(G/N)|=|\pi(G)|-1$, then N is a regular Sylow-subgroup of G. Thus, by the Schur-Zassenhaus theorem, N has a complement in G, denoted H. Let M be an extremely large subgroup of G containing H. Clearly, M does not contain N. In summary, it is known by calculation that there are at least $|\pi(G)|+p$ extremely large subgroups of G.

If G/N is an idempotent zero group, then the Sylow $t-$ subgroup of G is regular. Let $r=\max\{q\in\pi(G)|G$ Sylow $q-$ subgroups of G be regular in G} and $R\leq O_r(G)$ be a minimal regular subgroup of G. By the above discussion it can be assumed that G/R is power zero. At this point, G/N has at least $\pi(G/R)$ an extremely large subgroup. Thus G has at least $\pi(G/R)$ an extremely large subgroup containing R extremely large subgroups.

Let $q=\max\{q\in\pi(G)|G$ the Sylow $q-$ subgroup of G is not regular in G}. Taking $Q\in Syl_q(G)$, then $RQ($ G and $G=RN_G(Q)$ by Frattini assertion. Let M be an extremely large subgroup of G containing $N_G(Q)$. If $r<q$, then q is the largest prime in $\pi(G)$ and by Theorem 6 there are at least $q+1$ extremely large subgroups of G containing $N_G(Q)$ Obviously, none of these extremely large subgroups of G contains R. Thus, $|m(G)|\geq|\pi(G/N)|+q+1\geq|\pi(G)|+p$. If $r>q$, then M has at least r conjugate subgroups in G by $|G:M|=|R:R\cap M|\geq r$. Thus, $|m(G)|\geq|\pi(G/R)|+r\geq|\pi(G)|+p$. The theorem is proved.

Note that if the finite group G is power zero and $G=P_1\times P_2\times\cdots\times P_n, P_i\in Syl_{p_i}(G)$. Then $|m(G)|=|m(P_1)|+|m(P_2)|+\cdots+|m(P_n)|$. However, $|m(G)|\geq|m(P_1)|+|m(P_2)|+\cdots+|m(P_n)|+p$ does not necessarily hold when the finite group G is not idempotent zero, where $p=\min\{q\in\pi(G)|G$ of the Sylow $q-$-subgroups is not formal in G}. For example, if $G=A_4, P_2$ and P_3 are Sylow2- and 3-subgroups of G, respectively. At this point, $|m(P_2)|+|m(P_3)|+p=3+1+3=7$ and $|m(G)|=5$. We have the following corollary:

Corollary 4: Let G be a finite nonsolvable group and p be a minimal prime factor of $|G|$. Then the number of extremely large subgroups $|m(G)|\geq|\pi(G)|+p$.

III. B. Numerical simulation analysis

The numerical simulations in this chapter are based on the proposed non-commutative extremely large subgroup critical exponent computation method, which simulates a realistic network given the vertices with iteratively generated random graphs to obtain the desired interval estimates.

We first construct a model to observe this estimation.

Example 3: Generate a graph of 1200 random blocks with 120 vertices, where every vertex in each graph is uniformly selected from set $\{a,b,c,d,e\}$. Also the probability of connection between different blocks is represented by the following matrix:

$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} \frac{5}{6} & 0 & 0 & 0 & 0 \\ 0 & \frac{4}{6} & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{6} & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{6} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{6} \end{pmatrix} \end{array} \tag{50}$$

The expected value of the edge density of the above randomized block graph can be calculated to be 0.0489.

The simulation of the edge density of the randomized block model with 120 vertices whose block set is $\{a,b,c,d,e\}$ is shown in Fig. 1. As can be seen from the box-and-line plot, the mean value of $|E_n|/n^2$ is about 0.049 and its main perturbation range is the interval [0.036,0.063]. This artificial simulation demonstrates well the correctness of the theoretical results in this paper. When the probability of $|E_n|/n^2$ is between [0.036,0.063], its probability value converges to 1 exponentially as the number of graph vertices increases.

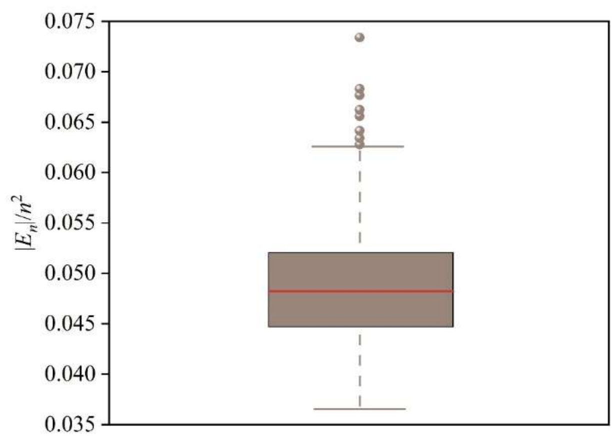

Figure 1: Edge density simulation of a random block model with 100 vertices

Next we select three real networks to see how this estimation is applied in practice.

Example 4: The data and objects described for these three types of networks are shown in Table 1.

Table 1: Descriptions of three real networks

	Points	Number of communities	Degree of fit
Karate club	36	2	0.068
Dolphin-like fall	64	2	0.041
Classification of political commentary books	112	3	0.041

In order to describe a realistic network in terms of a random block graph, it is first necessary to estimate the probability of connection between blocks, and below we give the strategy taken to estimate the probability of connection using sample data.

Connection probability of two edges in the same random block a:

$$\hat{p}_a = \frac{2m_a}{n_a(n_a-1)} \tag{51}$$

where m_a represents the number of edges in the block a, while n represents the corresponding number of vertices in this block. The probability of connecting two edges in any two different random blocks a and b:

$$\hat{p}_a = \frac{m_{ab}}{n_a n_b} \tag{52}$$

At this point m_{ab} denotes the number of edges between two blocks a and b.

Next we will generate 1200 random graphs for each of the three aforementioned realistic networks for simulation based on the data in Table 1.

(1) The data for the first network comes from a karate club containing 36 members from a university in the U.S. Two community networks exist, which are simulated using the random block model. The results of the random block simulation of the karate club are shown in Fig. 2, and the mean value of $|E_n|/n^2$ is obtained to be 0.0682, and its main fluctuation range is [0.047,0.084], which makes it possible to estimate the relationship of the people in the karate club accordingly.

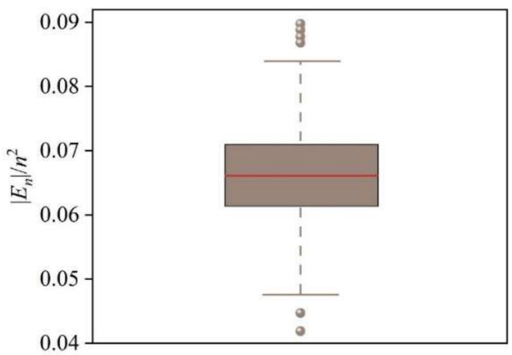

Figure 2: Random block simulation of the karate club community network

(2) The second real network is modeled from a study of dolphin communities. Explained in the language of graphs, the vertices of which are 64 broad-snouted dolphins from the Phantom Sound Channel in New Zealand, and the 161 edges in the graph indicate the paired dolphins in the community, and the related study, by observing these dolphins for a few years, found that they were divided into two communities according to different leaders, and simulated them through the dense random block graph, and the simulation of the dolphin community's random blocks is shown in Fig. 3, which can be obtained that the average value of $|E_n|/n^2$ is 0.0413.

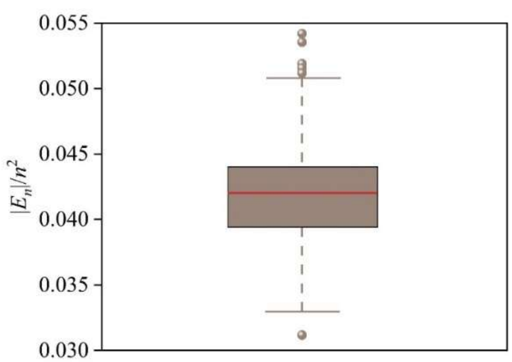

Figure 3: Random block simulation of dolphin communities

(3) The data for the third graph is taken from a study about online purchases of political books, where the nodes of the network represent 112 American political books, and the statistics for the number of edges are taken from Amazon.com's book sales, and represent two books that are often paired together to represent two books that are often purchased together, and which are contained roughly within the three community structures of liberals, conservatives, and moderates. A stochastic block simulation of categorized purchases of political books is shown in Figure 4. The expected value of its edge density $|E_n|/n^2$ is perturbed around 0.041, and the reasonableness of the critical exponent calculation method for non-exchangeable very large subgroups proposed in this paper can be seen with reference to Fig. 4.

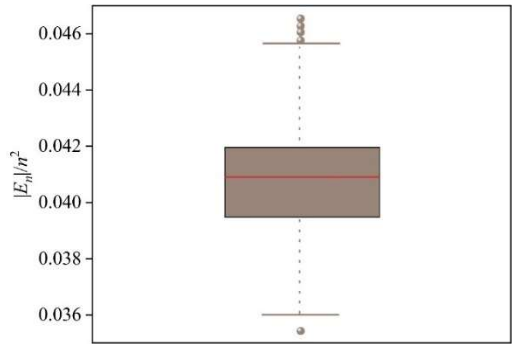

Figure 4: Random block simulation of classified purchase of political commentary books

Ⅳ. Conclusion

The number and distribution characteristics of non-exchangeable maximal subgroups of finite groups can be effectively analyzed by the critical index calculation method of non-exchangeable maximal subgroups proposed in this paper. The experimental results show that in the karate club network, the critical index calculated using this method is 0.0682 and fluctuates within the interval of [0.047, 0.084], which can accurately reflect the network relationship among group members. The simulation results of the dolphin group network similarly show that the expected value of the method is 0.0413 and fluctuates between [0.036, 0.063], which verifies the applicability of the method in different types of networks.

In addition, the simulation results in the political book classification network further demonstrate the effectiveness of the method in complex networks with an expected value of 0.041 and fluctuating in the interval of [0.037, 0.045]. These results show that the proposed method is able to capture the non-exchange features within the group better when dealing with real complex networks, thus providing a strong support for the classification and structural analysis of groups.

Overall, the method in this paper not only provides a new computational tool for the study of non-exchangeable extremely large subgroups, but also provides a theoretical basis for the classification and structural analysis of finite groups, as well as the modeling of actual networks. Future research can further optimize the method to improve the efficiency of its application in large-scale networks and explore its potential applications in other mathematical fields.

Funding

The author, Xueqing. Qin is supported by the promotion project of basic ability for young and middle-aged teachers in Universities Province of Guangxi (2024KY1726), Scientific research start-up fund project of Guilin Institute of Information Technology (XJ202077).

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